Revealed preferences over risk and uncertainty

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Abstract: We develop a nonparametric procedure, called the *lattice method*, for testing the consistency of contingent consumption data with a broad class of models of choice under risk and under uncertainty. Our method allows for risk loving and elation seeking behavior and can be used to calculate, via Afriat's efficiency index, the magnitude of violations from a particular model of choice. We evaluate the performance of different models (including expected utility, disappointment aversion, rank dependent utility, mean-variance utility, and stochastically monotone utility) in the data collected by Choi *et al.* (2007), in terms of pass rates, power, and predictive success.

Keywords: expected utility, rank dependent utility, disappointment aversion, Bronars power, predictive success, generalized axiom of revealed preference, first order stochastic dominance, mean-variance utility, Afriat's efficiency index

JEL classification numbers: C14, C60, D11, D12, D81

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1. Introduction

This paper is a contribution to the empirical investigation of decision making under risk and under uncertainty. At least since the famous paradox of Allais (1953), there has been a large literature developing models of choice under risk or under uncertainty that seek to give a better account of observed behavior than the expected utility (EU) model. An empirical literature that tests the EU and other models on experimental or field data has also emerged alongside these theoretical developments. These experiments often employ elicitation procedures in which subjects are in effect making repeated choices between two risky or uncertain outcomes; the data obtained in this way consist of a finite number of *binary choices*, which can then be used to partially recover a subject's preferences.

A more recent strand of experiments employs a different elicitation procedure, which we shall call the *budgetary choice* procedure. In this case subjects are asked to choose a preferred option from a potentially infinite set of alternatives. For example, a subject could be presented with a portfolio problem where she has to allocate her budget between two assets with state-contingent payoffs. Examples of experiments of this type include Loomes (1991), Gneezy and Potters (1997), Choi *et al.* (2007), Bayer *et al.* (2013), Choi *et al.* (2014), Ahn *et al.* (2014), Hey and Pace (2014), Cappelen *et al.* (2015), and Halevy, Persitz, and Zrill (2016).[1] The contribution of this paper is to develop and implement an empirical method that could be used to analyze data collected from portfolio decisions; it is applicable to experimental data where a budgetary choice elicitation procedure is employed and also to suitable field data. Our method allows us to test nonparametrically whether a data set is consistent with the EU model or some of its generalizations.

It is worth noting that models of decision making over time (such as the discounted utility model), as well as models of decision making involving both time and risk, are formally very similar to the EU model and its generalizations. In such contexts, budgetary choice procedures are also increasingly used in experiments and the empirical method developed in this paper could be used with little or no modification to test those models as well.[2]

[1] For an account of the advantages of such an approach, see Choi *et al.* (2007).

[2] See Andreoni and Sprenger (2012) and, for a comprehensive list of papers employing such procedures, Imai and Camerer (2016).

1.1 Testing revealed preferences on a finite lattice

One motivation for using the budgetary choice procedure is that it provides more information than a collection of binary choices, since a subject who chooses one bundle from a budget has revealed to the observer a preference for that bundle over an infinite number of alternatives in the budget set. This feature is also the reason why an empirical method is required to evaluate data collected from a budgetary procedure, whereas no such method is necessary for binary choices. Indeed, suppose we make a finite number of observations, where at observation t a subject chooses a lottery that gives a monetary payoff x_s^t in state s over one that gives y_s^t in state s (for $s = 1, 2, \ldots, \bar{s}$), where the probability of state s is known to be $\pi_s > 0$. Imagine that we would like to test if this data set is consistent with the EU model. Ignoring the issue of errors for the time being, checking for consistency with the EU model simply involves finding a strictly increasing Bernoulli function $u : \mathbb{R}_+ \to \mathbb{R}$ such that $\sum_{s=1}^{\bar{s}} \pi_s u(x_s^t) \geqslant \sum_{s=1}^{\bar{s}} \pi_s u(y_s^t)$ holds at every observation t. This amounts to solving a finite set of linear inequalities, and it is computationally straightforward to ascertain if a solution exists. However, it is clear that this method is no longer applicable when choices are instead made from a classical budget sets at every observation t, since even a single observed choice from a budget set reveals an infinite set of binary preferences.

The empirical method that we develop for solving this problem is very simple, and it is worth giving a short explanation here. Consider a data set with three observations and two states, as depicted in Figure 1. The subject chooses the contingent consumption bundle $(2, 5)$ from budget set B^1, $(6, 1)$ from B^2, and $(4, 3)$ from B^3. Assuming that the probability of state s is commonly known to be π_s, consistency with the EU model would require the existence of a strictly increasing Bernoulli function u such that $\pi_1 u(2) + \pi_2 u(5) \geqslant \pi_1 u(x) + \pi_2 u(y)$ for all (x, y) in B^1, and similarly at the other two observations.

In our main methodological result (Theorem 1), we show that this data set can be rationalized by the EU model if it can be rationalized on an appropriately modified consumption set. Specifically, let $\mathcal{X}$ be the set of consumption levels that are observed to have been chosen at some observation and in some state, plus zero; in this example $\mathcal{X} = \{0, 1, 2, 3, 4, 5, 6\}$. Then for the data set to be EU-rationalizable, it is *sufficient* (and obviously necessary) for it to be EU-rationalizable on the reduced consumption set $\mathcal{X}^2$, i.e., there is an increasing

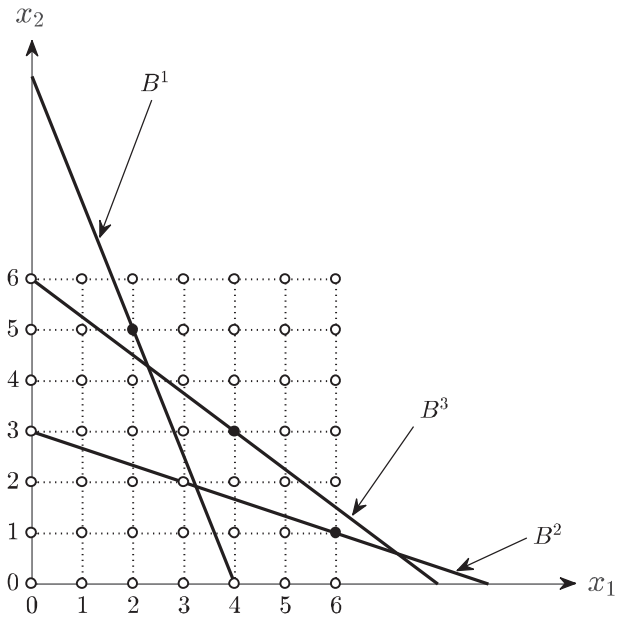


Figure 1: Constructing the finite lattice

function $\bar{u} : \mathcal{X} \to \mathbb{R}$ such that the expected utility of $(2, 5)$ is greater than any other bundle in $B^1 \cap \mathcal{X}^2$, and so forth. The set $\mathcal{X}^2$ is a finite lattice, depicted by the open circles in Figure 1. Therefore, checking for EU-rationalizability involves checking if there is a solution to a *finite* set of linear inequalities, a problem which is computationally feasible.[3]

This lattice method turns out to be very flexible: it can be used not just to check for EU-rationalizability, but also for consistency with other models of choice under risk (such as the rank dependent utility (RDU) model (Quiggin, 1982)) and under uncertainty (such as the maxmin expected utility model (Gilboa and Schmeidler, 1989)). The basic idea is always to convert an infinite collection of revealed preference pairs into a finite number involving only bundles on a finite lattice. Note also that the method *does not require linear budget sets*: it works for any type of constraint set, so long as it is compact.

1.2 Empirical implementation and findings

We implement our empirical method on a data set obtained from the portfolio choice experiment in Choi *et al.* (2007). Our objective is twofold: first, we wish to demonstrate that the lattice method actually works at a practical level and second, we are interested

[3] This example is EU-rationalizable on $\mathcal{X}^2$ and thus EU-rationalizable. One solution is $\bar{u}(0) = 0$, $\bar{u}(1) = 1$, $\bar{u}(2) = 4$, $\bar{u}(3) = 6$, $\bar{u}(4) = 7$, $\bar{u}(5) = 8$, and $\bar{u}(6) = 9$.

in what an analysis of these data with our methods will tell us about the performance of different models of choice under risk. In this experiment, each subject was asked to purchase Arrow-Debreu securities under different budget constraints. There were two states of the world, and it was commonly known that states occurred either symmetrically (each with probability 1/2) or asymmetrically (one with probability 1/3 and the other with probability 2/3). In their analysis, Choi *et al.* (2007) first checked whether a subject's observations were consistent with the maximization of a locally nonsatiated utility (LNU) function by applying Afriat's (1967) Theorem. Those subjects who passed or came sufficiently close to passing the test were then fitted to a parametric version of the disappointment aversion (DA) model (Gul, 1991). When there are two states, the DA model is a special case of the RDU model.

The lattice method developed in this paper, along with other recent advances to revealed preference techniques, make it possible for us to evaluate other models of choice under risk (apart from the LNU model) using a completely nonparametric approach. We can test if subjects pass the EU, DA, and RDU models using the lattice method. We also test if the subjects' choices are consistent with the maximization of a utility function that is *stochastically monotone* (the SMU model), in the sense that if a bundle dominates another with respect to first order stochastic dominance, then it must have higher utility; a test for the SMU model has recently been developed by Nishimura, Ok, and Quah (2017), and we implement it here for the first time. It is often considered desirable for a model of choice under risk to respect first order stochastic dominance; any such model, and this includes EU, DA and RDU, are special cases of the SMU model (which is in turn more stringent than the LNU model).

An important model of choice under risk which hitherto has not been subjected to a revealed preference test is the mean-variance utility (MVU) model (Markowitz, 1952). In this case, the utility of a contingent consumption bundle depends *solely* on the mean and variance of the state contingent payoff; this model can be tested by mapping contingent consumption bundles into mean-variance space and then testing the transformed data set using Forges and Minelli's (2009) generalization of Afriat's Theorem. The MVU model is, in theory, neither more nor less general than the EU model, though in the case with two goods, any MVU-rationalizable data set is also SMU-rationalizable.

With 50 observations collected on each subject, it is unsurprising that hardly any subject would be exactly rationalizable by even the LNU model. It is possible to quantify a data set's departure from a particular model of rationality using the *critical cost efficiency index* (Afriat (1972, 1973)); this index is widely used in the empirical revealed preference literature, including Choi *et al.* (2007). This index runs from 1 to 0, with the index equal to 1 if the data set passes the test exactly. We adopt the same measure of rationality in our empirical study. Given a data set from a *particular subject*, and for each of the models considered in this paper, it is possible to calculate this index; in the case of the EU, DA, and RDU models, this calculation again relies upon the lattice method.

In comparing the performance of different models, it is necessary to go beyond comparing (approximate) pass rates since a very permissive model will have a very high pass rate but also little predictive power. A standard way of measuring power in the empirical revealed preference literature is to estimate the probability of a randomly generated data set failing the test for a given model (Bronars, 1987); a model has high power if this probability is high. When one is investigating nested models, it is also natural to examine *relative power*: for example, if we randomly select a data set that is consistent with the LNU model, what is the probability that it is also consistent with the EU model? Even though the power of different models of choice under risk have been investigated in other contexts (see, for example, Harless and Camerer (1994) and Hey and Orme (1994)), we provide the first systematic investigation of this issue in the context of budgetary choice data.

So models can be compared in at least two dimensions: their pass rates and their power. There are different plausible ways of combining them into a single index; one possibility is to evaluate the difference between the pass rate and the model's *precision*, which is the probability of a random data set passing the test for the model (in other words, 1 minus the power). Selten (1991) provides an axiomatization of this index and calls it the *index of predictive success*. We evaluate the performance of different models according to this index in our study.

The following is a summary of our empirical findings:

- At a cost efficiency threshold of 0.9, more than 80% of subjects are consistent with the LNU model (summing across both the symmetric and asymmetric treatments).

- Among this group of subjects (who exhibit basic rationality), more than half are rationalizable by the EU model, with subjects under the symmetric treatment having a distinctly higher pass rate than those under the asymmetric treatment.

- The SMU and RDU models explain a sizable proportion of subjects whose behavior is not captured by the EU model. This is not true of the DA model, even though it is in principle a more permissive model than EU. The MVU model does not perform better than the EU model either.

- If we randomly generate an LNU-rationalizable data set, the probability of it being consistent with the RDU model, and hence the more stringent DA and EU models, is effectively zero. In other words, the power of these models is close to perfect, even among subjects who are consistent with the LNU model. If a data set is chosen randomly from those which are SMU-rationalizable, the probability of it being consistent with the RDU model remains very low, though it is no longer vanishingly small.

- All models have explanatory power in the sense that their indices of predictive success significantly exceed 0. This is true even if the index is calculated after conditioning on LNU-rationalizability. Furthermore, the EU, DA, and RDU models continue to have explanatory power even after conditioning on SMU-rationalizability.

- After conditioning on LNU-rationalizability, the SMU and RDU models have the highest indices of predictive success. These two models perform well because they manage to capture significantly more of the population than the EU model, without sacrificing power.

There is a large empirical literature that evaluates the performance of different models of choice under risk using experimental or field data (for example, Harless and Camerer (1994), Hey and Orme (1994), Bruhin, Fehr-Duda, and Epper (2010), and Barseghyan *et al.* (2013)). As we have already pointed out, the distinctive contribution of our empirical study is that it gives a completely nonparametric analysis of data collected through a budgetary elicitation procedure: no parametric assumptions are made on either Bernoulli function or, in the case of the RDU model, on the cumulative probability weighting function.

Our results appear to be broadly in line with the findings obtained in earlier studies, even though the very different methods employed across these studies make a formal comparison difficult. Other authors have also found that the RDU model performs well (see, for example, Bruhin, Fehr-Duda, and Epper (2010) and Barseghyan *et al.* (2013) and their references). We find that the EU model captures a significant portion of the subjects, though by no means everyone; this is broadly consistent with the fairly common finding that the EU model puts in a respectable performance (see, for example, Hey and Orme (1994)). The pass rate we report for the EU model is higher than that in some other papers (for example, Bruhin, Fehr-Duda, and Epper (2010) reports a pass rate of 20% for the EU model), but it is worth bearing in mind that our formulation of the EU model is about as permissive as it could get. We require the Bernoulli function to be increasing in money, but it is estimated nonparametrically and with no curvature assumptions (such as concavity). So in a sense we have given the EU model the greatest possible scope to perform well.

1.3 Relationship with the revealed preference literature

Our paper is related to the revealed preference literature originating from Afriat's (1967) Theorem, which characterizes consumer demand observations that are consistent with the maximization of an LNU function (see also Diewert (1973) and Varian (1982)). Afriat's Theorem has a *theoretical* significance in the sense that its intuitive behavioral characterization of LNU-rationalizability (through the *generalized axiom of revealed preference* (GARP)) provides a justification for that model in the consumer demand context, but it also provides a viable *empirical* method for testing LNU-rationalizability, which is why it has given rise to a large empirical literature.

A natural follow up to Afriat's contribution is to characterize those data sets that are rationalizable by more specialized utility functions. Among these papers are those which characterize observations of contingent consumption demand that are consistent with the EU model and (in more recent papers) some of its generalizations; these include Varian (1983a, 1983b, 1988), Green and Srivastava (1986), Diewert (2012), Echenique and Saito (2015), Chambers, Liu, and Martinez (2016), and Chambers, Echenique, and Saito (2016).[4]

[4] There is also a closely related literature on recovering expected utility from asset or contingent consumption demand *functions*, where, in effect the data set is assumed to be infinite (see, for example, Dybvig and Polemarchakis (1981) and Kubler, Selden, and Wei (2014)).

The principal difference between our results and this literature is that we do not rely on the methods of convex optimization; this means, in particular, that we do not require (or guarantee) the concavity of the Bernoulli function, and our results are applicable to data sets with general constraint sets rather than just linear budget sets. For reasons which we make clear later in the paper, the fact that we allow for nonlinear constraint sets means that our method can also be used to calculate Afriat's efficiency index.

It is worth mentioning that not all revealed preference results (involving budgetary observations) that flow from Afriat's Theorem have the feature of providing both theoretical insight and an empirical method. There are papers where the emphasis is on providing a characterization that offers theoretical insight; in other cases the emphasis is on providing an empirically viable method of model testing.[5] Our main methodological result (Theorem 1) says that, for a broad class of models, to check for rationalizability it suffices to check for rationalizabilty *as if* the subject's consumption space is some finite lattice (constructed from the data). By itself, the result does not furnish us with any theoretical motivation for one model or another; its principal value is in justifying an empirical method.

1.4 Organization of the paper

Section 2 provides a description of the lattice method and explains how it can be used to test the EU, DA, and RDU models in a budgetary choice environment. Further applications of the lattice method, including to models of decision making under uncertainty can be found in the Online Appendix. In Section 3 we explain Afriat's efficiency index and how the lattice method can be used to calculate this index. Section 4 describes the revealed preference tests for the LNU, SMU, and MVU models. The empirical application to the portfolio choice data collected by Choi *et al.* (2007) can be found in Section 5. Section 6 concludes.

[5] For example, Chambers, Liu, and Martinez (2016) provides a characterization of the former type, and they point out this distinction clearly in the introduction to their paper.

2. The lattice method

We assume that there is a finite set of states, denoted by $S = \{1, 2, \ldots, \bar{s}\}$. The contingent consumption space is $\mathbb{R}^{\bar{s}}_+$; for a typical consumption bundle $\mathbf{x} \in \mathbb{R}^{\bar{s}}_+$, the sth entry, x_s, specifies the consumption level in state s.[6] There are T observations in the data set $\mathcal{O} = \{(\mathbf{x}^t, B^t)\}_{t=1}^T$; by this we mean that the agent is observed choosing the bundle $\mathbf{x}^t$ from $B^t \subset \mathbb{R}^{\bar{s}}_+$. We assume that B^t is compact and that $\mathbf{x}^t \in \partial B^t$, where ∂B^t denotes the *upper boundary* of B^t.[7] The most important example of B^t is the classical linear budget set under complete markets, i.e.,

$$B^t = \{\mathbf{x} \in \mathbb{R}^{\bar{s}}_+ : \mathbf{p}^t \cdot \mathbf{x} \leqslant \mathbf{p}^t \cdot \mathbf{x}^t\}, \tag{1}$$

with $\mathbf{p}^t \gg 0$ denoting the vector of state prices. In this case, we may also write the data set as $\mathcal{O} = \{(\mathbf{x}^t, \mathbf{p}^t)\}_{t=1}^T$. The experiment conducted by Choi *et al.* (2007), whose data we analyze in Section 5, involves subjects choosing from linear budget sets.

Bear in mind, however, that our formulation only requires B^t to be compact and, in particular, it does not have to be a linear budget set. A crucial application requiring B^t to be nonlinear is found in Section 3, where we define approximate rationalizability. Another natural example of a nonlinear budget set is when a subject chooses contingent consumption through a portfolio of securities in an incomplete market; in this case, the budget set will be compact so long as the security prices do not admit arbitrage.[8]

Let $\{\phi(\cdot, t)\}_{t=1}^T$ be a collection of functions, where $\phi(\cdot, t) : \mathbb{R}^{\bar{s}}_+ \to \mathbb{R}$ is continuous and strictly increasing.[9] The data set $\mathcal{O} = \{(\mathbf{x}^t, B^t)\}_{t=1}^T$ is said to be *rationalizable by* $\{\phi(\cdot, t)\}_{t=1}^T$

[6] Our results do depend on the realization in each state being one-dimensional (which can be interpreted as a monetary payoff, but not a bundle of goods). This case is the one most often considered in applications and experiments and is also the assumption in a number of recent papers, including Kubler, Selden, and Wei (2014), Echenique and Saito (2015), and Chambers, Echenique, and Saito (2016). The papers by Varian (1983a, 1983b), Green and Srivastava (1986), Bayer *et al.* (2013), and Chambers, Liu, and Martinez (2016) allow for multi-dimensional realizations but (like the three aforementioned papers) they also require the convexity of the agent's preference over contingent consumption and linear budget sets.

[7] An element $\mathbf{y} \in B^t$ is in ∂B^t if there is no $\mathbf{x} \in B^t$ such that $\mathbf{x} > \mathbf{y}$. (For the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\bar{s}}$, we write $\mathbf{x} \geqslant \mathbf{y}$ if $x_s \geqslant y_s$ for all s, and $\mathbf{x} > \mathbf{y}$ if $\mathbf{x} \geqslant \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$. If $x_i > y_i$ for all i, we write $\mathbf{x} \gg \mathbf{y}$.) For example, if $B^t = \{(x, y) \in \mathbb{R}^2_+ : (x, y) \leqslant (1, 1)\}$, then $(1, 1) \in \partial B^t$ but $(1, 1/2) \notin \partial B^t$.

[8] Indeed, there is $\mathbf{p}^t \gg 0$ such that $B^t = \{\mathbf{x} \in \mathbb{R}^{\bar{s}}_+ : \mathbf{p}^t \cdot \mathbf{x} \leqslant \mathbf{p}^t \cdot \mathbf{x}^t\} \cap \{Z + \boldsymbol{\omega}\}$, where Z is the span of assets available to the subject and $\boldsymbol{\omega}$ is the subject's endowment of contingent consumption. Both B^t and $\mathbf{x}^t$ will be known to the observer, if he knows the asset prices, the subject's holding of securities, the asset payoffs in every state, and the subject's endowment of contingent consumption $\boldsymbol{\omega}$.

[9] By strictly increasing, we mean that $\phi(\mathbf{z}, t) > \phi(\mathbf{z}', t)$ if $\mathbf{z} > \mathbf{z}'$.

if there exists a continuous and strictly increasing function $u : \mathbb{R}_+ \to \mathbb{R}_+$, which we shall refer to as the *Bernoulli function*, such that

$$\phi(\mathbf{u}(\mathbf{x}^t), t) \geqslant \phi(\mathbf{u}(\mathbf{x}), t) \text{ for all } \mathbf{x} \in B^t, \tag{2}$$

where $\mathbf{u}(\mathbf{x}) = (u(x_1), u(x_2), \ldots, u(x_{\bar{s}}))$. In other words, $\mathbf{x}^t$ maximizes $\phi(\mathbf{u}(\mathbf{x}), t)$ in B^t. It is natural to require u to be strictly increasing since we typically interpret its argument to be money. The requirements on u guarantee that $\phi(\mathbf{u}(\cdot), t)$ is continuous and strictly increasing in $\mathbf{x}$. Note that continuity is an important property because it guarantees that the agent's utility maximization problem always has a solution on a compact constraint set.[10]

Expected utility. This model clearly falls within the framework we have set up. Indeed, suppose that both the observer and the agent know that the probability of state s at observation t is $\pi_s^t > 0$. If the agent is maximizing expected utility (EU),

$$\phi(u_1, u_2, \ldots, u_{\bar{s}}, t) = \sum_{s=1}^{\bar{s}} \pi_s^t u_s, \tag{3}$$

and (2) requires that

$$\sum_{s=1}^{\bar{s}} \pi_s^t u(x_s^t) \geqslant \sum_{s=1}^{\bar{s}} \pi_s^t u(x_s) \text{ for all } \mathbf{x} \in B^t, \tag{4}$$

i.e., the expected utility of $\mathbf{x}^t$ is greater than that of any other bundle in B^t. When there exists a Bernoulli function u such that (4) holds, we say that the data set is *EU-rationalizable with the probability weights* $\{\boldsymbol{\pi}^t\}_{t=1}^T$, where $\boldsymbol{\pi}^t = (\pi_1^t, \pi_2^t, \ldots, \pi_{\bar{s}}^t)$.

If $\mathcal{O}$ is rationalizable by $\{\phi(\cdot, t)\}_{t=1}^T$, then since the objective function $\phi(\mathbf{u}(\cdot), t)$ is strictly increasing in $\mathbf{x}$, the rationalizability condition (2) could be strengthened to

$$\phi(\mathbf{u}(\mathbf{x}^t), t) \geqslant \phi(\mathbf{u}(\mathbf{x}), t) \text{ for all } \mathbf{x} \in \underline{B}^t, \tag{5}$$

where $\underline{B}^t$ is the *downward extension of* B^t, i.e.,

$$\underline{B}^t = \{\mathbf{y} \in \mathbb{R}_+^{\bar{s}} : \mathbf{y} \leqslant \mathbf{x} \text{ for some } \mathbf{x} \in B^t\}. \tag{6}$$

Furthermore, the inequality in (5) is strict whenever $\mathbf{x} \in \underline{B}^t \backslash \partial \underline{B}^t$ (where $\partial \underline{B}^t$ refers to the upper boundary of $\underline{B}^t$). We define $\mathcal{X} = \{x' \in \mathbb{R}_+ : x' = x_s^t \text{ for some } t,\, s\} \cup \{0\}$; besides zero,

[10] The existence of a solution is obviously important if we are to make out-of-sample predictions. More fundamentally, a hypothesis that an agent is choosing a utility-maximizing bundle implicitly assumes that the utility function is such that an optimum exists for a reasonably broad class of constraint sets.

$\mathcal{X}$ contains those levels of consumption that are chosen at some observation and in some state. Since the data set is finite, so is $\mathcal{X}$. Given $\mathcal{X}$, we may construct $\mathcal{L} = \mathcal{X}^{\bar{s}}$, which consists of a finite grid of points in $\mathbb{R}^{\bar{s}}_+$; in formal terms, $\mathcal{L}$ is a finite lattice. Let $\bar{u} : \mathcal{X} \to \mathbb{R}_+$ be the restriction of the Bernoulli function u to $\mathcal{X}$. Given our observations, the following must hold:

$$\phi(\bar{\mathbf{u}}(\mathbf{x}^t), t) \geqslant \phi(\bar{\mathbf{u}}(\mathbf{x}), t) \text{ for all } \mathbf{x} \in \underline{B}^t \cap \mathcal{L} \text{ and} \tag{7}$$

$$\phi(\bar{\mathbf{u}}(\mathbf{x}^t), t) > \phi(\bar{\mathbf{u}}(\mathbf{x}), t) \text{ for all } \mathbf{x} \in \left(\underline{B}^t \backslash \partial \underline{B}^t\right) \cap \mathcal{L}, \tag{8}$$

where $\bar{\mathbf{u}}(\mathbf{x}) = (\bar{u}(x_1), \bar{u}(x_2), \ldots, \bar{u}(x_{\bar{s}}))$. Our main theorem says that the converse is also true.[11]

THEOREM 1. *Suppose that for some data set $\mathcal{O} = \{(\mathbf{x}^t, B^t)\}_{t=1}^T$ and collection of continuous and strictly increasing functions $\{\phi(\cdot, t)\}_{t=1}^T$, there is a strictly increasing function $\bar{u} : \mathcal{X} \to \mathbb{R}_+$ that satisfies conditions (7) and (8). Then there is a Bernoulli function $u : \mathbb{R}_+ \to \mathbb{R}_+$ that extends $\bar{u}$ and guarantees the rationalizability of $\mathcal{O}$ by $\{\phi(\cdot, t)\}_{t=1}^T$.*[12]

What Theorem 1 achieves is *domain reduction*: checking the rationalizability of $\mathcal{O}$ is equivalent to checking rationalizability in the case where the agent's consumption space is considered to be $\mathcal{L}$ rather than $\mathbb{R}^{\bar{s}}_+$, which (crucially) reduces the rationality requirements to a *finite* number of inequalities involving the observed choice and an alternative (see (7) and (8)), and with the Bernoulli function defined on $\mathcal{X}$ rather than $\mathbb{R}_+$.

The intuition for Theorem 1 ought to be strong. Given $\bar{u}$ satisfying (7) and (8), we can define the step function $\hat{u} : \mathbb{R}_+ \to \mathbb{R}_+$ where $\hat{u}(r) = \bar{u}([r])$, with $[r]$ being the largest element of $\mathcal{X}$ weakly lower than r, i.e., $[r] = \max\{r' \in \mathcal{X} : r' \leqslant r\}$. Notice that $\phi(\hat{\mathbf{u}}(\mathbf{x}^t), t) = \phi(\bar{\mathbf{u}}(\mathbf{x}^t), t)$ and, for any $\mathbf{x} \in \underline{B}^t$, $\phi(\hat{\mathbf{u}}(\mathbf{x}), t) = \phi(\bar{\mathbf{u}}([\mathbf{x}]), t)$, where $[\mathbf{x}] = ([x_1], [x_2], \ldots, [x_{\bar{s}}])$ in $\underline{B}^t \cap \mathcal{L}$. Clearly, if $\bar{u}$ obeys (7) and (8) then $\mathcal{O}$ is rationalized by $\{\phi(\cdot, t)\}_{t=1}^T$ and $\hat{u}$ (in the

[11] Note that $\underline{B}^t$ cannot be replaced with B^t in (7) and (8). For example, suppose there are two observations, where $\mathbf{x}^1 = (1, 0)$ is chosen from $B^1 = \{(x_1, x_2) \in \mathbb{R}^2_+ : 2x_1 + x_2 = 2\}$ and $\mathbf{x}^2 = (0, 1)$ is chosen from $B^2 = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 + 2x_2 = 2\}$. This pair of observations cannot be rationalized by any increasing utility function (even though the 'budget sets' are just lines) and, in particular, cannot be rationalized in the sense of Theorem 1 (with ϕ constant across t). However, since $\mathcal{L} = \{(0,0), (0,1), (1,0), (1,1)\}$, $B^1 \cap \mathcal{L} = \{(1,0)\}$ and $B^2 \cap \mathcal{L} = \{(0,1)\}$, so conditions (7) and (8) are vacuous. On the other hand $\left(\underline{B}^1 \backslash \partial \underline{B}^1\right) \cap \mathcal{L}$ contains $(0,1)$ and $\left(\underline{B}^2 \backslash \partial \underline{B}^2\right) \cap \mathcal{L}$ contains $(1,0)\}$, so (8) requires $\phi(\bar{\mathbf{u}}(\mathbf{x}^1)) > \phi(\bar{\mathbf{u}}(\mathbf{x}^2))$ and $\phi(\bar{\mathbf{u}}(\mathbf{x}^1)) < \phi(\bar{\mathbf{u}}(\mathbf{x}^2))$, which plainly cannot happen. This allows us to conclude, correctly, that the data set is not rationalizable.

[12] The increasing assumptions on ϕ and $\bar{u}$ ensure that we may confine ourselves to checking (7) and (8) for undominated elements of $\underline{B}^t \cap \mathcal{L}$, i.e., $\mathbf{x} \in \underline{B}^t \cap \mathcal{L}$ such that there does not exist $\mathbf{x}' \in \underline{B}^t \cap \mathcal{L}$ with $\mathbf{x} < \mathbf{x}'$.

sense that (2) holds). This falls short of the claim in the theorem only because $\hat{u}$ is neither continuous nor strictly increasing; the proof in the Appendix shows how one could in fact construct a Bernoulli function with these additional properties.

2.1 Testing the expected utility model

Theorem 1 provides us with a very convenient way of testing EU-rationalizability. The theorem tell us that $\mathcal{O} = \{(\mathbf{x}^t, B^t)\}_{t=1}^T$ is EU-rationalizable with the probability weights $\{\boldsymbol{\pi}^t\}_{t=1}^T$ if and only if there is a collection of real numbers $\{\bar{u}(r)\}_{r\in\mathcal{X}}$ such that

$$0 \leqslant \bar{u}(r') < \bar{u}(r) \text{ whenever } r' < r, \tag{9}$$

and the inequalities (7) and (8) hold, where $\phi(\cdot, t)$ is defined by (3). This is a linear program and it is both formally *solvable* (in the sense that there is an algorithm that can decide within a known number of steps whether or not there is a solution to this set of linear inequalities) and also computationally feasible.

At this point it is worth emphasizing that requiring a data set to be EU-rationalizable is certainly more stringent than simply requiring it to be rationalizable by a locally nonsatiated utility function. Indeed, while a data set with a single observation $(\mathbf{x}^t, \mathbf{p}^t)$ must be consistent with the maximization of a strictly increasing (hence locally nonsatiated) utility function, even a single observation can be incompatible with the EU model.

Example 1. Suppose that there are two equiprobable states of the world, and at the price vector $\mathbf{p}^t = (p_1^t, p_2^t)$ such that $p_1^t > p_2^t$, the agent purchases a bundle $\mathbf{x}^t$ such that $x_1^t > x_2^t$. We claim that this is not EU-rationalizable, or in other words, the agent cannot buy strictly more of the more expensive good. Indeed, such an observation is not compatible with the maximization of any symmetric and strictly increasing utility function on $\mathbb{R}_+^2$: with symmetry, the bundle (y_1, y_2), where $y_1 = x_2^t$ and $y_2 = x_1^t$, is strictly cheaper than $\mathbf{x}^t$ but gives the same utility, so $\mathbf{x}^t$ is not optimal. Such an observation will also fail the lattice test, since (y_1, y_2) is in $\mathcal{L} \cap (\underline{B}^t \backslash \partial \underline{B}^t)$ but the condition (8) is not satisfied. $\square$

On the other hand, our test is strictly less stringent than a test of EU-rationalizability that also requires the Bernoulli function to be concave (such as Green and Srivastava (1986)); imposing concavity on the Bernoulli function has observable implications over and above those which flow simply from the EU model, as the following example demonstrates.

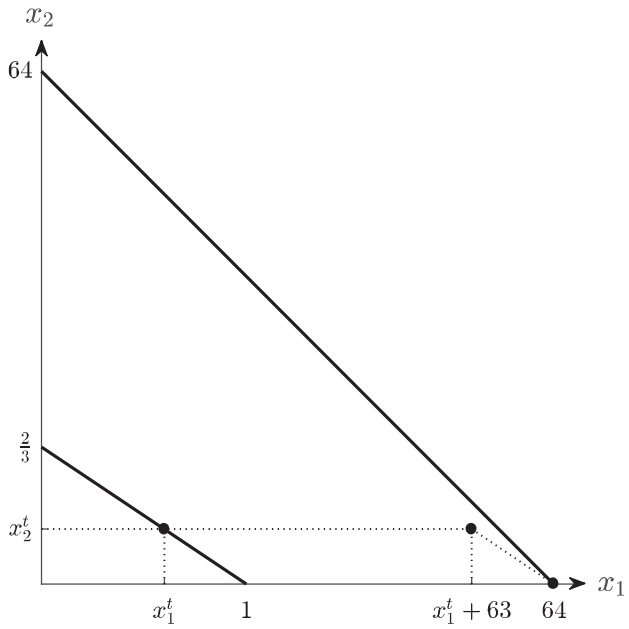


Figure 2: EU-rationalizable but not concave EU-rationalizable data set

Example 2. Suppose an agent maximizes expected utility and has the Bernoulli function $u(y) = (y-4)^3$, which is strictly concave for $y < 4$ and strictly convex otherwise.[13] There are two states of the world, which occur with equal probability. At $\mathbf{p}^t = (1, 3/2)$ and with wealth 1, the agent chooses $x_1 \in [0, 1]$ to maximize $f(x_1) = (x_1 - 4)^3 + [2(1 - x_1)/3 - 4]^3$. Over this range, the Bernoulli function is strictly concave and so is f; one could check that $f'(1) < 0$ so that there is unique interior solution which we denote $\mathbf{x}^t$ (see Figure 2).[14] At the prices $\mathbf{p}^{t'} = (1, 1)$ with wealth equal to 64, the agent chooses $x_1 \in [0, 64]$ to maximize $g(x_1) = (x_1 - 4)^3 + (60 - x_1)^3$. It is straightforward to check that g is strictly convex on $[0, 64]$ and it is thus maximized at the two end points $(0, 64)$ and $(64, 0)$.

Now consider a data set consisting of two observations: the bundle $\mathbf{x}^t$ chosen at $\mathbf{p}^t = (1, 3/2)$ and $\mathbf{x}^{t'} = (64, 0)$ chosen at $\mathbf{p}^{t'} = (1, 1)$. This data set is EU-rationalizable and it will pass the lattice test, but it cannot be rationalized by a concave Bernoulli function. Indeed,

$$u(x_1^t) + u(x_2^t) \geqslant u(1) + u(0) \tag{10}$$

since $(1, 0)$ is affordable to the agent when $\mathbf{x}^t$ is chosen. If we further assume that u is concave,

[13] A Bernoulli function with a concave region followed by a convex region was used by Friedman and Savage (1948, Figure 2) to explain why an agent could simultaneously buy insurance and accept risky gambles.

[14] Solving the (quadratic) first order condition gives $\mathbf{x}^t \approx (0.83, 0.11)$.

$u(1) - u(x_1^t) \geqslant u(64) - u(x_1^t + 63)$; substituting this into (10), we obtain $u(x_1^t + 63) + u(x_2^t) \geqslant u(64) + u(0)$. The bundle $((x_1^t + 63), x_2^t)$ is strictly cheaper than $\mathbf{x}^{t'} = (64, 0)$ at $\mathbf{p}^{t'}$ (see Figure 2), so $\mathbf{x}^{t'}$ cannot be optimal. To conclude, while $\mathcal{O} = \{(\mathbf{p}^t, \mathbf{x}^t), (\mathbf{p}^{t'}, \mathbf{x}^{t'})\}$ is indeed EU-rationalizable, it is not EU-rationalizable with a concave Bernoulli function. $\square$

We have shown that a data set is EU-rationalizable if and only if it is EU-rationalizable on $\mathcal{L}$ and the latter is in turn equivalent to the existence of a function $\bar{u}$ obeying conditions (7), (8), and (9) (with $\phi(\mathbf{u}, t) = \sum_{s=1}^{\bar{s}} \pi_s^t u_s$). Conditions (7) and (8) generate a finite list of preference pairs between some chosen bundle $\mathbf{x}^t$ and another bundle $\mathbf{x}$ in $\underline{B}^t \cap \mathcal{L}$ or $\left(\underline{B}^t \backslash \partial \underline{B}^t\right) \cap \mathcal{L}$. Condition (9) can also be reformulated as saying that the bundle $(r, r, \ldots, r)$ is strictly preferred to $(r', r', \ldots, r')$ whenever $r > r'$, for r, $r' \in \mathcal{X}$. We gather these together in a list $\{(\mathbf{a}^j, \mathbf{b}^j)\}_{j=1}^M$, where for all $j \leqslant N$ (with $N < M$), the bundle $\mathbf{a}^j$ is weakly preferred to $\mathbf{b}^j$ (so the pairs are drawn from (7)) and for $j > N$, $\mathbf{a}^j$ is strictly preferred to $\mathbf{b}^j$ (so the pairs are drawn from (8) and (9)). Each bundle $\mathbf{a}^j$ can be written in its lottery form $\hat{\mathbf{a}}^j$, where $\hat{\mathbf{a}}^j$ is the vector with $|\mathcal{X}|$ entries, with the ith entry giving the probability of ith ranked number in $\mathcal{X}$; similarly, $\mathbf{b}^j$ can be written in its lottery form $\hat{\mathbf{b}}^j$. For example, in the example given in the introduction, $\mathcal{X} = \{0, 1, 2, 3, 4, 5, 6\}$ and the two states are equiprobable, so the bundle $(2, 5)$ chosen from B^1 has the lottery form $(0, 0, 1/2, 0, 0, 1/2, 0)$.

We know from Fishburn (1975) that the list $\{(\mathbf{a}^j, \mathbf{b}^j)\}_{j=1}^M$ is rationalizable by EU (i.e., there is $\bar{u}$ that solves (7), (8), and (9) with $\phi(\mathbf{u}, t) = \sum_{s=1}^{\bar{s}} \pi_s^t u_s$) if and only if there does not exist λ^j with $\sum_{j=1}^M \lambda^j = 1$, $\lambda^j \geqslant 0$ for all j, and $\lambda^j > 0$ for some $j > N$, such that

$$\sum_{j=1}^{M} \lambda^j \hat{\mathbf{a}}^j = \sum_{j=1}^{M} \lambda^j \hat{\mathbf{b}}^j. \tag{11}$$

This condition is very intuitive: assuming that the agent has a preference over lotteries, the independence axiom says that the lottery $\sum_{j=1}^M \lambda^j \hat{\mathbf{a}}^j$ must be strictly preferred to $\sum_{j=1}^M \lambda^j \hat{\mathbf{b}}^j$, and therefore (11) is excluded.[15] Put another way, a violation of Fishburn's condition must imply a violation of the independence axiom.

To summarize, we have shown that *a data set* $\mathcal{O} = \{(\mathbf{x}^t, B^t)\}_{t=1}^T$ *is EU-rationalizable*

[15] To be precise, suppose that the agent has a preference over lotteries with prizes in $\mathcal{X}$. The independence axiom says that if lottery $\hat{\mathbf{a}}$ is preferred (strictly preferred) to $\hat{\mathbf{b}}$, then $\gamma\hat{\mathbf{a}} + (1-\gamma)\hat{\mathbf{c}}$ is preferred (strictly preferred) to $\gamma\hat{\mathbf{b}} + (1-\gamma)\hat{\mathbf{c}}$, where $\hat{\mathbf{c}}$ is another lottery and $\gamma \in [0, 1]$. Repeated application of this property and the transitivity of the preference will guarantee that $\sum_{j=1}^M \lambda^j \hat{\mathbf{a}}^j$ is strictly preferred to $\sum_{j=1}^M \lambda^j \hat{\mathbf{b}}^j$.

with probability weights $\{\boldsymbol{\pi}^t\}_{t=1}^T$ *if and only if it is EU-rationalizable with probability weights* $\{\boldsymbol{\pi}^t\}_{t=1}^T$ *on the domain* $\mathcal{L}$ *and this in turn holds if and only if the preference pairs on* $\mathcal{L}$ *(as revealed by the data) do not contain a contradiction of the independence axiom of the form (11).* Examples 3 and 5, to be explained later in the paper, give examples of data sets which violate Fishburn's condition on $\mathcal{L}$ and are hence not EU-rationalizable.

2.2 Other applications of the lattice method

So far, we have considered tests of EU-rationalizability in the case where the probability of each state is known to both the agent and the observer. The testing procedure could be extended to the case where no objective probabilities can be attached to each state. A data set $\mathcal{O} = \{(\mathbf{x}^t, B^t)\}_{t=1}^T$ is *rationalizable by subjective expected utility* (SEU) if there exist probability weights $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_{\bar{s}}) \gg 0$, with $\sum_{s=1}^{\bar{s}} \pi_s = 1$, and a Bernoulli function $u : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $t = 1, 2, \ldots, T$,

$$\sum_{s=1}^{\bar{s}} \pi_s u(x_s^t) \geqslant \sum_{s=1}^{\bar{s}} \pi_s u(x_s) \text{ for all } \mathbf{x} \in B^t. \tag{12}$$

In this case, ϕ is independent of t and instead of being fixed, it is required to belong to the family of functions Φ_{SEU} such that $\phi \in \Phi_{SEU}$ if $\phi(\mathbf{u}) = \sum_{s=1}^{\bar{s}} \pi_s u_s$ for some $\boldsymbol{\pi} \gg 0$. By Theorem 1, the data set $\mathcal{O} = \{(\mathbf{x}^t, B^t)\}_{t=1}^T$ can be rationalized by some $\phi \in \Phi_{SEU}$ if *there is a strictly increasing* $\bar{u}$ *such that (7) and (8) holds* and it is clear that these conditions are also necessary. These conditions form a system of bilinear inequalities with unknowns $\{\pi_s\}_{s=1}^{\bar{s}}$ and $\{\bar{u}(r)\}_{r \in \mathcal{X}}$. Even though solving a bilinear problem may be computationally intense, the Tarski-Seidenberg Theorem tells us that the problem is nonetheless solvable.

For many of the standard models of decision making under risk or uncertainty, the rationalizability problem has a structure similar to that of SEU in the sense that rationalizability by a particular model involves finding a Bernoulli function u and a function ϕ belonging to some family Φ that rationalize the data, and this problem can in turn be transformed (via Theorem 1) into a problem of solving a system of bilinear inequalities. In the Online Appendix, we show how tests for various models (for example, maxmin expected utility (Gilboa and Schmeidler, 1989) or choice acclimating personal equilibrium (Köszegi and Rabin, 2007) can be devised using on Theorem 1).

Rank dependence and disappointment aversion. The rank dependent utility (RDU) model (Quiggin, 1982) is a prominent model of choice under risk. In Section 5, we report the findings of a test of this model, so we explain it here in greater detail. Let $\pi_s > 0$ be the objective probability of state s.[16] Given a contingent consumption bundle $\mathbf{x}$, we can rank the entries of $\mathbf{x}$ from the smallest to the largest, with ties broken by the rank of the state. We denote by $r(\mathbf{x}, s)$, the rank of x_s in $\mathbf{x}$. For example, if there are five states and $\mathbf{x} = (1, 4, 4, 3, 5)$, we have $r(\mathbf{x}, 1) = 1$, $r(\mathbf{x}, 2) = 3$, $r(\mathbf{x}, 3) = 4$, $r(\mathbf{x}, 4) = 2$, and $r(\mathbf{x}, 5) = 5$. A rank dependent expected utility function gives to the bundle $\mathbf{x}$ the utility $V(\mathbf{x}) = \sum_{s=1}^{\bar{s}} \delta(\mathbf{x}, s) u(x_s)$ where $u : \mathbb{R}_+ \to \mathbb{R}$ is a Bernoulli function,

$$\delta(\mathbf{x}, s) = g\left(\sum_{\{s' : r(\mathbf{x}, s') \leqslant r(\mathbf{x}, s)\}} \pi_{s'}\right) - g\left(\sum_{\{s' : r(\mathbf{x}, s') < r(\mathbf{x}, s)\}} \pi_{s'}\right), \tag{13}$$

and $g : [0, 1] \to \mathbb{R}$ is a continuous and strictly increasing function. (If $\{s' : r(\mathbf{x}, s') < r(\mathbf{x}, s)\}$ is empty, we let $g\left(\sum_{\{s' : r(\mathbf{x}, s') < r(\mathbf{x}, s)\}} \pi_{s'}\right) = g(0)$.) The function g distorts the cumulative distribution of the bundle $\mathbf{x}$, so that an agent maximizing rank dependent utility can behave as though the probability he attaches to a state depends on the relative attractiveness of the outcome in that state. Since u is strictly increasing, $\delta(\mathbf{x}, s) = \delta(\mathbf{u}(\mathbf{x}), s)$ and therefore $V(\mathbf{x}) = \phi(\mathbf{u}(\mathbf{x}))$, where for any vector $\mathbf{u} = (u_1, u_2, \ldots, u_{\bar{s}})$,

$$\phi(\mathbf{u}) = \sum_{s=1}^{\bar{s}} \delta(\mathbf{u}, s) u_s.$$

Note that the function ϕ is continuous and strictly increasing in $\mathbf{u}$. So we have shown that V has the form assumed in Theorem 1 and we could use that result to devise a test for RDU-rationalizability.

We discuss the multiple state version of the test in the Appendix; at this point it suffices to explain the two-state case, which is the one relevant to the implementation reported in Section 5. Let $\rho_s = g(\pi_s)$ be the distorted value of π_s (the true probability of state s, for $s = 1, 2$). Then $\phi(u_1, u_2) = \rho_1 u_1 + (1 - \rho_1) u_2$ if $u_1 \leqslant u_2$ and $\phi(u_1, u_2) = (1 - \rho_2) u_1 + \rho_2 u_2$ if $u_1 > u_2$; by Theorem 1, a sufficient (and obviously necessary) condition for a data set $\mathcal{O} = \{(\mathbf{x}^t, B^t)\}_{t=1}^T$ to be RDU-rationalizable is for there to be a solution to (7) and (8), with this formula for ϕ. This test involves solving a set of inequalities that are bilinear in the unknowns

[16] To keep the notation light, we confine ourselves to the case where $\boldsymbol{\pi}$ does not vary across observations. There is no conceptual difficulty in allowing for this.

$\{\bar{u}(r)\}_{r\in\mathcal{X}}$ and $\{\rho_1, \rho_2\}$. While solving bilinear inequalities can in general be computationally demanding, this case can be handled with relative ease: in our implementation, we simply let ρ_1 and ρ_2 take different values on a very fine grid in $[0,1]^2$, subject to $\rho_1 \leqslant \rho_2$ (if and only if $\pi_1 \leqslant \pi_2$) and perform the associated linear test.

It is worth mentioning at this point that for certain values of ρ_1 and ρ_2, the function ϕ is clearly not concave or even quasiconcave, and therefore we cannot guarantee the quasiconcavity of the agent's utility over contingent consumption, *even if* we restrict ourselves to concave Bernoulli functions. While the lattice method still works in these cases, it is not possible to formulate a test for rationalizability that allows for non-quasiconcave utility functions using concave optimization methods (such as those cited in Section 1.3) because the first order conditions are no longer *sufficient* for optimality.

We also implement a lattice test of Gul's (1991) model of disappointment aversion (DA). When there are two states, the DA model is a special case of the RDU model in which an additional restriction is placed on ρ_1 and ρ_2. Specifically, there is some $\beta \in (-1, \infty)$ such that, for $s = 1, 2$,

$$\rho_s = \frac{(1+\beta)\pi_s}{1+\pi_s\beta}. \tag{14}$$

Note that this restriction has bite only if $\rho_1 \neq \rho_2$, so in fact the RDU and DA models are identical when $\pi_1 = \pi_2$. If $\beta = 0$ the agent simply maximizes expected utility. If $\beta > 0$, we have $\rho_s > \pi_s$, so the agent attaches a probability on s that is higher than the objective probability when s is the less favorable state; in this case, the agent is said to be *disappointment averse*. If $\beta < 0$, then $\rho_s < \pi_s$, and the agent is said to be *elation seeking*; this is an instance where the function ϕ is not quasiconcave. As in the RDU model, we test the DA model by letting β take on different values and performing the associated linear test.

While it is well known that the RDU and EU models lead to different predictions, it not immediately clear that they are observationally distinct in the context of observations drawn from linear budgets. We end this section with an example of a data set that is RDU-rationalizable but not EU-rationalizable.

Example 3. Suppose the data set consists of three observations $(\mathbf{x}^t, \mathbf{p}^t)$, for $t = 1, 2, 3$, where $\mathbf{p}^1 = (1, q)$, $\mathbf{x}^1 = (a, a)$; $\mathbf{p}^2 = (1, 1/q)$, $\mathbf{x}^2 = (b, b)$; and $\mathbf{p}^3 = (1, (1/q^2) + \epsilon)$, $\mathbf{x}^3 = (a + (a-b)/q, b + (b-a)q)$, with $q > 1$ and $a < b$, and $\epsilon > 0$ is a small number. The three

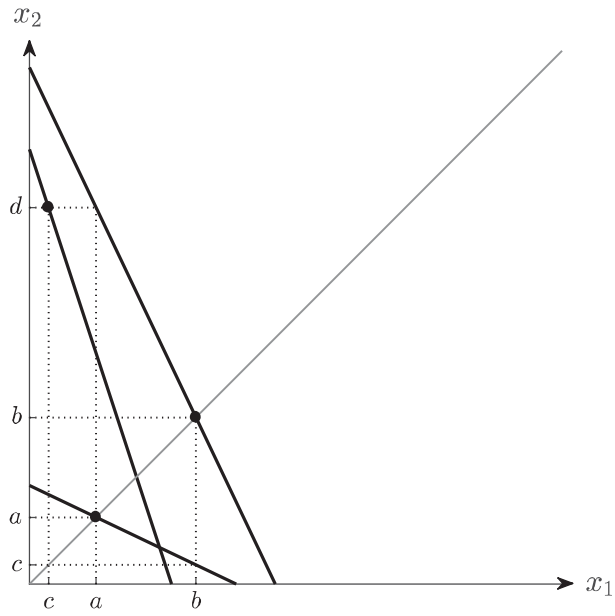


Figure 3: RDU-rationalizable but not EU-rationalizable data set

observations are depicted Figure 3, where $c = a + (a - b)/q$ and $d = b + (b - a)q$.

We claim that these observations are not EU-rationalizable if the two states are equiprobable. Suppose that they are, for some Bernoulli function u. Then the first observation tells us that $2u(a) \geqslant u(b) + u(c)$, since (b, c) is available when (a, a) is chosen. Similarly, from the second observation, we know that $2u(b) \geqslant u(a) + u(d)$. Together this gives

$$u(b) - u(d) \geqslant u(a) - u(b) \geqslant u(c) - u(a),$$

from which we obtain $u(a) + u(b) \geqslant u(c) + u(d)$. On the other hand, it is straightforward to check that, with $\epsilon > 0$, the bundle (a, b) is strictly cheaper than (c, d) at $\mathbf{p}^3$, which leads to a contradiction since (c, d) is chosen over (a, b) at the third observation.[17]

We claim that these observations are RDU-rationalizable; in fact, they can be rationalized with a smooth and concave Bernoulli function. Suppose $V(x_1, x_2) = \rho u(x_1) + (1 - \rho)u(x_2)$ when $x_1 \leqslant x_2$ and $V(x_1, x_2) = (1-\rho)u(x_1) + \rho u(x_2)$ when $x_1 > x_2$, with $\rho = q/(q+1)$. Since $\rho > 1/2$, the agent displays disappointment aversion. So long as u is strictly concave, the agent's utility is maximized at $x_1 = x_2$ whenever $p_1 = 1$ and $p_2 \in [1/q, q]$. So V rationalizes

[17] Equivalently, note that there is a violation of Fishburn's condition. The bundle/lottery (a, a) is preferred to (b, c), (b, b) is preferred to (a, c), and (c, d) is strictly preferred to (a, b). However, the compound lottery where (a, a), (b, b) and (c, d) each occur with probability 1/3 is stochastically equivalent to the compound lottery where (b, c), (a, d) and (a, b) each occur with probability 1/3.

the first two observations. To justify the third, it suffices to find u such that $u' > 0$ and $u'' < 0$ satisfying the first order condition

$$\frac{\rho u'(c)}{(1-\rho)u'(d)} = q\frac{u'(c)}{u'(d)} = \frac{p_1^3}{p_2^3} = \frac{q^2}{1+q^2\epsilon}.$$

If ϵ is sufficiently small, this is possible since the price ratio p_1^3/p_2^3 is greater than q.[18] $\square$

3. Goodness of fit

The revealed preference tests presented in the previous section are 'sharp', in the sense that a data set either passes the test for a given model or it fails. This either/or feature of the tests is not peculiar to our results but is true of all classical revealed preference tests, including Afriat's. It would, of course, be desirable to develop a way of measuring *the extent* to which a certain class of utility functions succeeds or fails in rationalizing a data set, and the most common approach adopted in the revealed preference literature to address this issue was developed by Afriat (1972, 1973) and Varian (1990).[19, 20] We now give an account of this approach and explain why implementing it in our setting is possible (or at least no more difficult than implementing the exact tests).

Suppose that the observer collects a data set $\mathcal{O} = \{(\mathbf{x}^t, B^t)\}_{t=1}^T$; following the earlier literature, we focus attention on the case where B^t is a classical linear budget set given by (1). For any number $e^t \in [0,1]$, we define

$$B^t(e^t) = \{\mathbf{x} \in \mathbb{R}_+^{\bar{s}} : \mathbf{p}^t \cdot \mathbf{x} \leqslant e^t \mathbf{p}^t \cdot \mathbf{x}^t\} \cup \{\mathbf{x}^t\}. \tag{15}$$

Clearly $B^t(e^t)$ is smaller than B^t and shrinks with the value of e^t. Let $\mathcal{U}$ be a collection of utility functions defined on $\mathbb{R}_+^{\bar{s}}$ belonging to a given family; for example, $\mathcal{U}$ could be the family of locally nonsatiated utility functions (which was the family considered by Afriat (1972, 1973) and Varian (1990) in their work). We define the set $E(\mathcal{U})$ in the following manner: a

[18] If it were smaller than q, this would not be possible since the concavity of u requires $u'(c) \geqslant u'(d)$.

[19] For examples where Afriat-Varian type indices are used to measure a model's fit, see Mattei (2000), Harbaugh, Krause, and Berry (2001), Andreoni and Miller (2002), Choi *et al.* (2007, 2014), Beatty and Crawford (2011), and Halevy, Persitz, and Zrill (2016). See also Echenique, Lee, and Shum (2011), which develops and applies a related index called the money pump index.

[20] For an account of why such measures may be more suitable then other measures of goodness-of-fit, such as the sum of squared errors between observed and predicted demands, see Varian (1990) and Halevy, Persitz, and Zrill (2016).

vector $\mathbf{e} = (e^1, e^2, \ldots, e^T)$ is in $E(\mathcal{U})$ if there is some function $U \in \mathcal{U}$ that rationalizes the modified data set $\mathcal{O}(\mathbf{e}) = \{(\mathbf{x}^t, B^t(e^t))\}_{t=1}^T$, i.e., $U(\mathbf{x}^t) \geqslant U(\mathbf{x})$ for all $\mathbf{x} \in B^t(e^t)$. Clearly, the data set $\mathcal{O}$ is rationalizable by a utility function in $\mathcal{U}$ if and only if the unit vector $(1, 1, \ldots, 1)$ is in $E(\mathcal{U})$. We also know that $E(\mathcal{U})$ must be nonempty since it contains the vector $\mathbf{0}$, and it is clear that if $\mathbf{e} \in E(\mathcal{U})$ then $\mathbf{e}' \in E(\mathcal{U})$, where $\mathbf{e}' < \mathbf{e}$. The closeness of the set $E(\mathcal{U})$ to the unit vector is a measure of how well the utility functions in $\mathcal{U}$ can explain the data. Afriat (1972, 1973) suggests measuring this distance with the supnorm, so the distance between $\mathbf{e}$ and 1 is $D_A(\mathbf{e}) = 1 - \min_{1 \leqslant t \leqslant T} \{e^t\}$, while Varian (1990) suggests that we choose the square of the Euclidean distance, i.e., $D_V(\mathbf{e}) = \sum_{t=1}^T (1 - e^t)^2$.

Measuring distance by the supnorm has the advantage that it is computationally straightforward, and it is also the measure most commonly used in the empirical revealed preference literature, so this is the approach that we adopt in our implementation (see Section 5). Note that $D_A(\mathbf{e}) = D_A(\tilde{\mathbf{e}})$ where $\tilde{\mathbf{e}}$ is the vector with identical entries equal to $\min\{e^1, e^2, \ldots, e^T\}$, where $\mathbf{e} = (e^1, e^2, \ldots, e^T)$. Since $\tilde{\mathbf{e}} \leqslant \mathbf{e}$, we obtain $\tilde{\mathbf{e}} \in E(\mathcal{U})$ whenever $\mathbf{e} \in E(\mathcal{U})$. Therefore, $\min_{\mathbf{e} \in E(\mathcal{U})} D_A(\mathbf{e}) = \min_{\mathbf{e} \in \tilde{E}\ \mathcal{U}}\ D_A\ \mathbf{e}$, where $E\ \mathcal{U}\qquad \mathbf{e}\quad E\ \mathcal{U}\ : e^t\quad e^{t'}$ for any $t, t\ \ $, i.e., in searching for $\mathbf{e}\quad E\ \mathcal{U}$ that minimizes the supnorm distance from $\ 1, 1, \ldots, 1\ $, we can focus our attention on the set $E\ \mathcal{U}\ $, which consists of those vectors in $E\ \mathcal{U}$ that shrink each observed budget set by the same proportion. Given a data set $\mathcal{O}\qquad \mathbf{x}^t, \mathbf{p}^t\ \ {}_t^T\ {}_1$, Afriat refers to $\sup\ e :\ e, e, \ldots, e\qquad E\ \mathcal{U}$ as the *critical cost efficiency index*; we say that $\mathcal{O}$ is *rationalizable in* $\mathcal{U}$ *at the efficiency index/threshold* e if $\ e\ , e\ , \ldots, e\qquad E\ \mathcal{U}\ .$

Suppose that for a given data set, the critical cost efficiency index is 0.95. In that case, while we cannot guarantee that $\mathbf{x}^t$ is optimal in the true budget set, we know that there is some utility function in $\mathcal{U}$ for which $\mathbf{x}^t$ is optimal in $B^t\ \ 0.95\ \ $ at every observation t. With this utility function, there could be bundles in B^t which the subject prefers to $\mathbf{x}^t$, but choosing such a bundle (instead of $\mathbf{x}^t$) will not lead to savings of more than 5%. Furthermore, this number is tight in the following sense: given any $\epsilon\quad 0$, then for every utility function in $\mathcal{U}$, there is at least one observation t where the subject could indeed have saved $\ 5\quad \epsilon\ \%$ of her expenditure.[21] We can interpret this index as a characteristic of the subject and, specifically, a measure of her bounded rationality; the bounded rationality could have arisen because she

[21] Formally, for every $U \in \mathcal{U}$ there is t such that $\max\{U(\mathbf{x}) : \mathbf{p}^t \cdot \mathbf{x} \leqslant \mathbf{p}^t \cdot \mathbf{x}^t(0.95 + 0.01\epsilon)\} > U(\mathbf{x}^t)$.

is simply incapable of better decision making, or it could be that she has consciously or otherwise judged that it is not, from a broader perspective, rational for her to expend the mental powers needed for exactly rational portfolio decisions.

Calculating the efficiency index (or, more generally, an index based on the Euclidean metric) will require checking whether a particular vector $\mathbf{e} = (e^1, e^2, \ldots, e^T)$ is in $E(\mathcal{U})$, i.e., whether $\{(\mathbf{x}^t, B^t(e^t))\}_{t=1}^T$ is rationalizable by a member of $\mathcal{U}$. When $\mathcal{U}$ is the family of all locally nonsatiated utility functions, Afriat (1972, 1973) provides a necessary and sufficient condition for the rationalizability of $\{(\mathbf{x}^t, B^t(e^t))\}_{t=1}^T$; we explain this in Section 4.

More generally, the calculation of the efficiency index will hinge on whether there is a suitable test for the rationalizability of $\{(\mathbf{x}^t, B^t(e^t))\}_{t=1}^T$ by members of $\mathcal{U}$. Even if a test of the rationalizability of $\{(\mathbf{x}^t, B^t)\}_{t=1}^T$ by members of $\mathcal{U}$ is available, this test may rely on the convexity or linearity of the budget sets B^t; in this case, extending the test so as to check the rationalizability of $\mathcal{O}(\mathbf{e}) = \{(\mathbf{x}^t, B^t(e^t))\}_{t=1}^T$ is not straightforward since the sets $B^t(e^t)$ are clearly nonconvex. Crucially, this is *not* the case with the lattice test, which is applicable even for nonconvex constraint sets. Thus extending our testing procedure to measure goodness of fit in the form of the efficiency index involves no additional difficulties.

3.1 Approximate smooth rationalizability

While Theorem 1 guarantees that there is a Bernoulli function u that extends $\bar{u} : \mathcal{X} \to \mathbb{R}_+$ and rationalizes the data when the required conditions are satisfied, the Bernoulli function is not necessarily smooth (though it is continuous and strictly increasing by definition). Of course, the smoothness of u is commonly assumed in applications of expected utility and related models and its implications can appear to be stark. For example, suppose that it is commonly known that states 1 and 2 occur with equal probability and we observe the agent choosing $(1, 1)$ at a price vector (p_1, p_2), with $p_1 \neq p_2$. This observation is incompatible with a smooth EU model; indeed, given that the two states are equiprobable, the slope of the indifference curve at $(1, 1)$ must equal -1 and thus it will not be tangential to the budget line and will not be a local optimum. On the other hand, it is trivial to check that this observation is EU-rationalizable in our sense. In fact, one could even find a *concave* Bernoulli function $u : \mathbb{R}_+ \to \mathbb{R}_+$ for which $(1, 1)$ maximizes expected utility. (Such a u will, of course, have a kink at 1.)

These two facts can be reconciled by noticing that, even though this observation cannot be exactly rationalized by a smooth Bernoulli function, it is in fact possible to find a smooth function that comes arbitrarily close to rationalizing it. Indeed, given any strictly increasing and continuous function u defined on a compact interval of $\mathbb{R}_+$, there is a strictly increasing and smooth function $\tilde{u}$ that is uniformly and arbitrarily close to u on that interval. As such, if a Bernoulli function $u : \mathbb{R}_+ \to \mathbb{R}_+$ rationalizes $\mathcal{O} = \{(\mathbf{x}^t, B^t)\}_{t=1}^T$ by $\{\phi(\cdot, t)\}_{t=1}^T$, then for any efficiency threshold $e \in (0, 1)$, there is a smooth Bernoulli function $\tilde{u} : \mathbb{R}_+ \to \mathbb{R}_+$ that rationalizes $\mathcal{O}' = \{(\mathbf{x}^t, B^t(e))\}_{t=1}^T$ by $\{\phi(\cdot, t)\}_{t=1}^T$. In other words, if a data set is rationalizable by some Bernoulli function, then it can also be rationalized by a smooth Bernoulli function, for any efficiency threshold arbitrarily close to 1. In this sense, imposing a smoothness requirement on the Bernoulli function does not radically alter a model's ability to explain a given data set.

4. Other revealed preference tests

In addition to the EU, DA, and RDU models, the testing of which is made possible by the lattice method described in Sections 2 and 3, there are other models of decision making under risk which we implement empirically in Section 5. In this section, we describe these models, their tests, and the relationships between them.

Locally nonsatiated utility. The locally nonsatiated utility (LNU) model is the most permissive of the models that we consider in the sense that all others are special cases. A utility function $U : \mathbb{R}_+^{\bar{s}} \to \mathbb{R}$ is *locally nonsatiated* if at every open neighborhood N of $\mathbf{x} \in \mathbb{R}_+^{\bar{s}}$, there is $\mathbf{y} \in N$ such that $U(\mathbf{y}) > U(\mathbf{x})$. A data set $\mathcal{O} = \{(\mathbf{x}^t, \mathbf{p}^t)\}_{t=1}^T$ is LNU-rationalizable if it can be rationalized by a continuous and locally nonsatiated utility function.

Afriat's Theorem tells us that $\mathcal{O}$ is LNU-rationalizable if and only if it obeys a consistency condition known as the *generalized axiom of revealed preference* (GARP).[22] Afriat (1972, 1973) also shows that there is natural generalization of GARP that characterizes LNU-rationalizability at some efficiency index e, which we now describe. Let $\mathcal{D} = \{\mathbf{x}^t : t =$

[22] This term and its acronym were coined by Varian (1982), who also provides a proof of Afriat's Theorem. To be specific, the theorem says that GARP is necessary whenever the data set is rationalizable by a locally nonsatiated utility function (continuity is not needed); conversely, when a data set obeys GARP, then it is rationalizable by a continuous, strictly increasing, and concave utility function.

$1, 2, \ldots, T\}$; in other words, $\mathcal{D}$ consists of those bundles that have been observed somewhere in the data set. For bundles $\mathbf{x}^t$ and $\mathbf{x}^{t'}$ in $\mathcal{D}$, $\mathbf{x}^t$ is said to be *revealed preferred to* $\mathbf{x}^{t'}$ at the efficiency index (or threshold) e (we denote this by $\mathbf{x}^t \succcurlyeq_e^* \mathbf{x}^{t'}$) if $\mathbf{x}^{t'} \in B^t(e)$, where $B^t(e)$ is given by (15);[23] $\mathbf{x}^t$ is said to be *strictly revealed preferred to* $\mathbf{x}^{t'}$ (and we denote this by $\mathbf{x}^t \succ_e^* \mathbf{x}^{t'}$) if $\mathbf{x}^{t'} \in B^t(e)$ and $\mathbf{p}^t \cdot \mathbf{x}^{t'} < e\, \mathbf{p}^t \cdot \mathbf{x}^t$. $\mathcal{O}$ is LNU-rationalizable at the efficiency index e if and only if, whenever there are observations $(\mathbf{p}^{t_i}, \mathbf{x}^{t_i})$ (for $i = 1, 2, \ldots, n$) in $\mathcal{O}$ satisfying

$$\mathbf{x}^{t_1} \succcurlyeq_e^* \mathbf{x}^{t_2},\ \mathbf{x}^{t_2} \succcurlyeq_e^* \mathbf{x}^{t_3},\ \ldots\ ,\ \mathbf{x}^{t_{n-1}} \succcurlyeq_e^* \mathbf{x}^{t_n},\ \text{and}\ \mathbf{x}^{t_n} \succcurlyeq_e^* \mathbf{x}^{t_1}, \tag{16}$$

then we cannot replace $\succcurlyeq_e^*$ with $\succ_e^*$ anywhere in this chain; in other words, while there can be revealed preference cycles in $\mathcal{O}$, they cannot contain a strict revealed preference. This property is a generalization of GARP, which is the special case where $e = 1$. Checking for this property is computationally undemanding: the (strict) revealed preference relations on $\mathcal{D}$ can be easily constructed; once this has been established, we can apply Warshall's algorithm to construct the transitive closure of the revealed preference relations and then check for the absence of cycles containing strict revealed preferences.

Stochastically monotone utility. For $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}_+^{\bar{s}}$, we write $\mathbf{x} \geqslant_{FSD} \mathbf{y}$ if $\mathbf{x}$ first order stochastically dominates $\mathbf{y}$ (given the payoffs and the objectively known probabilities) and write $\mathbf{x} >_{FSD} \mathbf{y}$ if $\mathbf{x} \geqslant_{FSD} \mathbf{y}$ and the two distributions are distinct. One way of sharpening the LNU model is to require that the utility function $U : \mathbb{R}_+^{\bar{s}} \to \mathbb{R}$ be *stochastically monotone*. By this we mean that $U(\mathbf{x}) > (\geqslant) U(\mathbf{y})$ whenever $\mathbf{x} >_{FSD} \mathbf{y}$ ($\mathbf{x} \geqslant_{FSD} \mathbf{y}$). Note that the RDU, DA, and EU models are all special cases of the SMU model.

In the Choi *et al.* (2007) experiment, there are only two states. With two states, it is straightforward to check that when $\pi_1 = \pi_2 = 1/2$, a utility function is stochastically monotone if and only if it is strictly increasing and symmetric; when $\pi_2 > \pi_1$, a utility function U is stochastically monotone if and only if it is strictly increasing and $U(a, b) > U(b, a)$ whenever $b > a$.

A data set $\mathcal{O} = \{(\mathbf{x}^t, \mathbf{p}^t)\}_{t=1}^T$ is said to be rationalizable by the stochastically monotone utility (SMU) model, or *SMU-rationalizable*, if there is a continuous and stochastically mono-

[23] Our terminology differs a little from the standard, which refers to $\succcurlyeq_e^*$ as the *direct revealed preference* relation and uses *revealed preference* to refer to the transitive closure of this relation. Since our exposition avoids any discussion of the transitive closure, we have adopted the simpler terminology here.

tone utility function U that rationalizes the observations. Since a utility function U that is stochastically monotone will be strictly increasing, it is also locally nonsatiated. Hence any SMU-rationalizable data set is also LNU-rationalizable but the converse is not true. Indeed, the single observation given in Example 1 must necessarily be LNU-rationalizable, but it cannot be rationalized by any symmetric and strictly increasing utility function.

Nishimura, Ok, and Quah (2017) have recently developed a test for SMU-rationalizability. The test can be thought of as a version of GARP, but with revealed preference relations defined in a way that is appropriate for the SMU model. We say that the bundle $\mathbf{x}^t$ *is SMU-revealed preferred to* $\mathbf{x}^{t'}$ at the efficiency threshold e (for $\mathbf{x}^t$ and $\mathbf{x}^{t'}$ in $\mathcal{D}$) if there is a bundle $\mathbf{y}$ such that $\mathbf{y} \in B^t(e)$ and $\mathbf{y} \geqslant_{FSD} \mathbf{x}^{t'}$; this revealed preference is *strict* if $\mathbf{y}$ can be chosen to satisfy $\mathbf{y} >_{FSD} \mathbf{x}^t$. Nishimura, Ok, and Quah (2017) show that a data set is SMU-rationalizable at threshold e if and only if it does not admit SMU-revealed preference cycles (such as (16)) containing strict SMU-revealed preferences; we call the latter property F-GARP (at the efficiency threshold e), where 'F' stands for first order stochastic dominance. Clearly this result is analogous to the characterization for LNU-rationalizability, except that the revealed preferences are defined differently. When there are two states, working out the SMU-revealed preference relations is straightforward, and therefore checking F-GARP is also easy to implement.

Mean-variance utility. Even though it does not have the broad axiomatic foundations of the expected utility model and some of its generalizations, the mean-variance utility (MVU) model (Markowitz, 1952) is widely used in finance. In this model the subject evaluates a contingent consumption bundle $\mathbf{x}$ solely on the mean, $\mu(\mathbf{x})$, and variance, $\sigma^2(\mathbf{x})$, of its payoffs; therefore, the utility of $\mathbf{x}$ is $U(\mathbf{x}) = F(\mu(\mathbf{x}), \sigma(\mathbf{x}))$ where the aggregator function F is required to be continuous, strictly increasing in the first argument, and strictly decreasing in the second. A data set $\mathcal{O} = \{(\mathbf{x}^t, \mathbf{p}^t)\}_{t=1}^T$ is MVU-rationalizable if we can find F having the required properties such that the utility function U rationalizes the observations.

Notice that the agent's utility is not necessarily strictly increasing in the mean-variance model. Given a bundle $\mathbf{x}$, it is clear that $\mathbf{y} = \mathbf{x} + (\epsilon, 0, 0, \ldots, 0) > \mathbf{x}$ if $\epsilon > 0$, and $\mathbf{y}$ will have a larger mean payoff than $\mathbf{x}$. However, it is possible for $\mathbf{y}$ to have a larger variance as well and an agent with a mean-variance utility could prefer $\mathbf{x}$ to $\mathbf{y}$. That said, a mean-variance

utility function is locally nonsatiated, so this model is a special case of the LNU model. Indeed, for any bundle $\mathbf{x}$, the bundle $\mathbf{x} + (\epsilon, \epsilon, \ldots, \epsilon)$ has a higher mean than $\mathbf{x}$ but the same variance, so its utility must be strictly higher than $\mathbf{x}$.

A revealed preference test of this model was first proposed by Varian (1983b), with the additional requirement that the aggregator function F be concave. The latter property is required in Varian's test since it is based on testing the first order conditions; for the same reason, Varian's test is also an exact test and it is not immediately clear how one could calculate the efficiency threshold at which a data set passes the test (in the event that it fails the exact test). The test was never implemented by Varian, and so far, this model has never been tested against data using a nonparametric method.

The approach we employ in testing the MVU model is based instead on the test developed by Forges and Minelli (2009); this approach has the advantage that it does not impose a concavity requirement on F and it can also be easily adapted to calculate the efficiency threshold at which a data set passes the test. Forges and Minelli (2009) propose a generalization of Afriat's Theorem where the budget sets are not necessarily linear; while they did not have the MVU model in mind, their result can be applied to test this model in the following manner. Let $M^t(e)$ be the image of $B^t(e)$ in mean-variance space, i.e.,

$$M^t(e) = \{(a, b) : \text{ there is } \mathbf{x} \in B^t(e) \text{ such that } \mu(\mathbf{x}) = a \text{ and } \sigma^2(\mathbf{x}) = b\}.$$

We say that $\mathbf{x}^t$ *is MVU-revealed preferred to* $\mathbf{x}^{t'}$ at the efficiency threshold e (for $\mathbf{x}^t$ and $\mathbf{x}^{t'}$ in $\mathcal{D}$) if there is a bundle in $M^t(e)$ with a higher mean and lower variance, i.e., there is $(a, b) \in M^t(e)$ such that $(a, -b) \geqslant (\mu(\mathbf{x}^{t'}), -\sigma^2(\mathbf{x}^{t'}))$; this revealed preference is said to be *strict* if $(a, -b) > (\mu(\mathbf{x}^{t'}), -\sigma^2(\mathbf{x}^{t'}))$. We say that a data set obeys MV-GARP (at the efficiency threshold e) if any MVU-revealed preference cycle of the form (16) cannot contain a strict revealed preference. It is straightforward to check that, if a data set $\mathcal{O}$ is MVU-rationalizable at the efficiency threshold e, then it must obey MV-GARP at that threshold. In fact, by applying Forges and Minelli (2009), we know that MV-GARP is also sufficient to guarantee that $\mathcal{O}$ is MVU-rationalizable at the efficiency threshold e.

Alone among the models we consider, MVU is not a generalization of the EU model. The two models are also observationally distinguishable, in the sense that there are data sets consistent with one model but not the other as the following examples demonstrate.

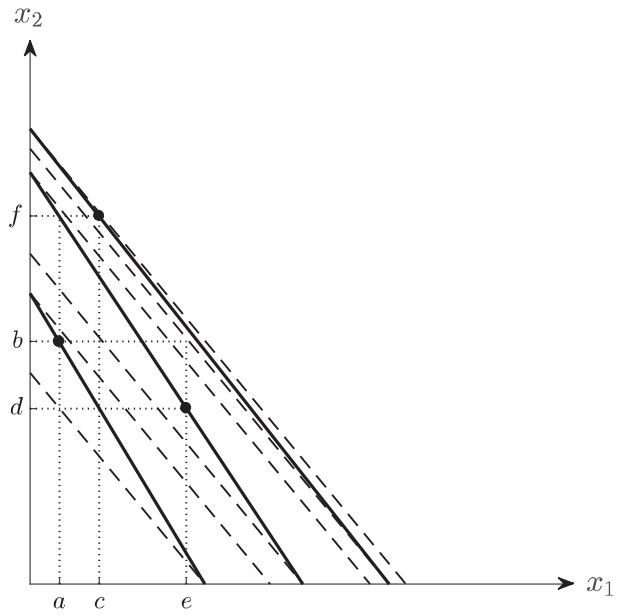


Figure 4: MVU-rationalizable but not EU-rationalizable data set

Example 4. Suppose $\pi_1 > \pi_2$ and at observation t, we have $p_1^t > p_2^t$, with $p_1^t/p_2^t < \pi_1/\pi_2$, and $x_1^t < x_2^t$. This observation is clearly compatible with the EU model, but we claim it is not MVU-rationalizable. Indeed, consider the bundle (a, a) on the budget line, so $p_1^t x_1^t + p_2^t x_2^t = p_1^t a + p_2^t a$. Then $\pi_1 x_1^t + \pi_2 x_2^t < \pi_1 a + \pi_2 a$ because $p_1^t/p_2^t < \pi_1/\pi_2$. Therefore the bundle (a, a) has a higher mean payoff than (x_1^t, x_2^t) and no variance, which means it must be strictly preferred to (x_1^t, x_2^t). It is also affordable at observation t, which means that (x_1^t, x_2^t) cannot be MVU-rationalized. □

Example 5. Figure 4 depicts three observations. At observations 1, 2, and 3, the chosen bundles are (a, b), (e, d), and (c, f) respectively. We claim that these observations are not compatible with the EU model. From the first observation, we obtain $\pi_1 u(a) + \pi_2 u(b) \geqslant \pi_1 u(c) + \pi_2 u(d)$. From the second, we obtain $\pi_1 u(e) + \pi_2 u(d) \geqslant \pi_1 u(a) + \pi_2 u(f)$. Together these imply that $\pi_1 u(e) + \pi_2 u(b) \geqslant \pi_1 u(c) + \pi_2 u(f)$, but this is contradicted by the third observation. On the other hand, we claim that these observations are MVU-rationalizable. Suppose π_1 and π_2 are such that the loci of bundles with the same mean are depicted by the dashed lines in the figure. Then any bundle on the budget line of the first observation has a lower mean than any bundle on the budget line of the second observation and, similarly, any bundle on the second budget line has a lower mean than any bundle on the budget line of

the third observation. This guarantees that it obeys MV-GARP, and so these observations are MVU-rationalizable. □

When there are just two states (as in the experimental data of Choi *et al.* (2007)), the MVU model is more stringent than the SMU model in the sense that any data set that is MVU-rationalizable at some efficiency threshold e will also be SMU-rationalizable at the same threshold.[24] However, in the case where the states are equiprobable, the converse is also true generically, in the sense that the converse holds whenever the data set does not contain observed prices where $p_1^t = p_2^t$.[25] In other words, when the two states are equiprobable, it is effectively impossible to separate the MVU and SMU models through an experiment with linear budget sets. The proofs for these claims can be found in the Online Appendix.

5. Implementation

We examine the data collected from the well known portfolio choice experiment in Choi *et al.* (2007). The experiment was performed on 93 undergraduate subjects at the University of California, Berkeley. Every subject was asked to make consumption choices on 50 decision problems under risk. The subject divided her budget between two Arrow-Debreu securities, with each security paying one token if the corresponding state was realized, and zero otherwise. In a symmetric treatment applied to 47 subjects, each state of the world occurred with probability 1/2, and in two asymmetric treatments applied to 17 and 29 subjects, the probability of the first state was 1/3 and 2/3, respectively. These probabilities were objectively known. Income was normalized to one, and state prices were chosen at random and varied across subjects.

In their analysis, Choi *et al.* (2007) first tested whether subjects were maximizing locally nonsatiated utility functions. Those subjects who pass GARP at a sufficiently high efficiency threshold were then fitted individually to a two-parameter version of the disappointment

[24] Even with two states, the set of mean-variance and stochastically monotone utility functions are not contained in one another. The claim made here is that when a data set is rationalizable by a mean-variance utility function, then it is also rationalizable by a stochastically monotone utility function, though the two functions need not be the same.

[25] This claim is false if we allow for $p_1^t = p_2^t$. Indeed, suppose that there is just one observation with $p_1^t = p_2^t$. Then *any* bundle on the budget line leads to an SMU-rationalizable observation but only the bundle on the budget line setting $x_1 = x_2$ is MVU-rationalizable since variance is uniquely minimized at that bundle, while all bundles on the budget line have the same mean consumption.

aversion model of Gul (1991). The lattice method developed in this paper, together with other revealed preference tests developed since Choi *et al.*'s paper (notably Forges and Minelli (2009) and Nishimura, Ok, and Quah (2017)), make it possible for us to re-analyze the same data using solely revealed preference techniques, without appealing to any parametric assumptions. In this section we evaluate different models of decision making with these newly-developed tests according to three criteria: (1) the ability of the model to explain the observed data; (2) the precision of the model's predictions (in various senses that we shall define); and (3) an index combining (1) and (2).

The models we consider are the locally nonsatiated utility (LNU), the stochastically monotone utility (SMU), mean-variance utility (MVU), rank dependent utility (RDU), disappointment aversion (DA), and expected utility (EU) models. Testing for the RDU, DA, and EU models applies the lattice method as explained in Sections 2 and 3. The LNU, SMU, and MVU models could be tested by checking GARP, F-GARP, and MV-GARP (as explained in Section 4). The reader should bear in mind that the LNU model contains all the other models. The SMU model nests the RDU model, which nests the DA model which nests the EU model. Observationally, the MVU model is also nested within the SMU model. When the two states are equiprobable the DA and RDU models are identical (see Section 2); furthermore, the MVU and SMU models are generically indistinguishable (in the sense described in Section 4). Therefore, with two equiprobable states, all models are nested within one another, with the EU model being the most stringent, followed (with increasing generality) by the RDU/DA, SMU/MVU, and LNU models.

5.1 Exact pass rates and distributions of efficiency indices

We first test all six models on the Choi *et al.* (2007) data, and the results from these tests are displayed in Table 1, where each cell contains a pass rate. Across 50 decision problems, 16 out of 93 subjects obey GARP and are therefore consistent with the LNU model; subjects in the symmetric treatment perform distinctly better than those in the asymmetric treatment. Of the 16 subjects who are LNU-rationalizable, only 4 are consistent with the SMU model. Still fewer subjects are rationalizable by the MVU, RDU, DA, and EU models.

Given that we observe 50 decisions for every subject, it may not be intuitively surprising that so many subjects should have violated GARP (let alone more stringent conditions). We

$\pi_1 = 1/2$		$\pi_1 \neq 1/2$		Total	
LNU	12/47 (26%)	LNU	4/46 (9%)	LNU	16/93 (17%)
SMU/MVU	1/47 (2%)	SMU	3/46 (7%)	SMU	4/93 (4%)
		MVU	0/46 (0%)	MVU	1/93 (1%)
RDU/DA	1/47 (2%)	RDU	2/46 (4%)	RDU	3/93 (3%)
		DA	1/46 (2%)	DA	2/93 (2%)
EU	1/47 (2%)	EU	1/46 (2%)	EU	2/93 (2%)

Table 1: Pass rates

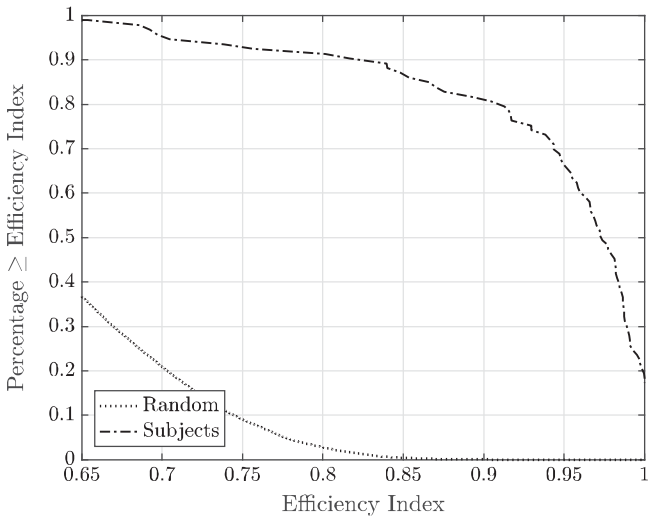
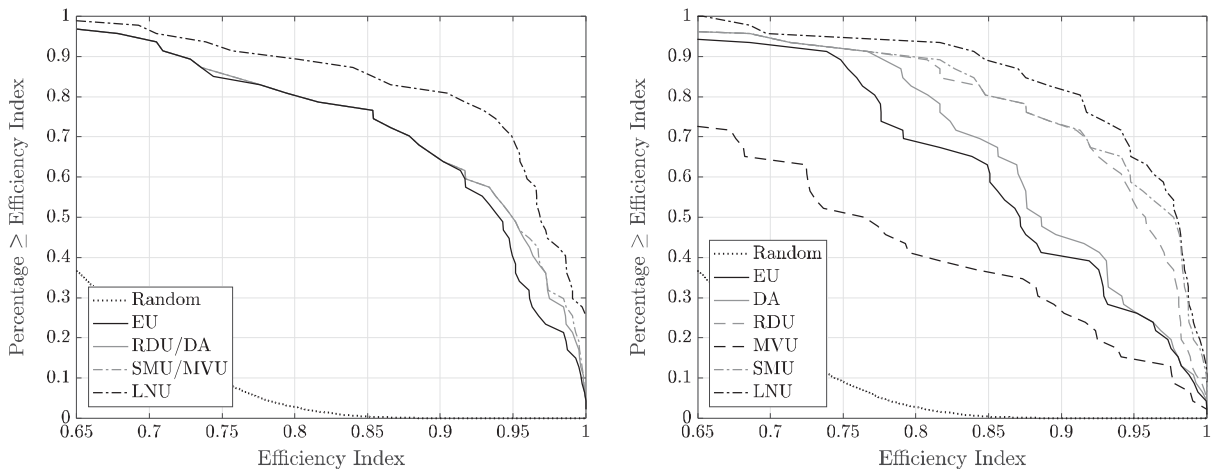


Figure 5: Distribution of efficiency indices for the LNU model

next investigate the efficiency thresholds at which subjects pass the different tests. First, we calculate the efficiency index at which each of the 93 subjects passes the LNU model; this empirical distribution is depicted in Figure 5. (Note that this figure is essentially a replication of Figure 4 in Choi *et al.* (2007).) We see that more than 80% of subjects have an efficiency index above 0.9, and more than 90% have an index above 0.8. A first glance at these results suggest that the data are largely consistent with the LNU model.

To better understand what the observed distribution of efficiency indices says about the success or failure of a particular model to explain the data collected, it is useful to see what distribution of efficiency indices will arise if we postulate an alternative form of behavior. We adopt an approach first suggested by Bronars (1987) that simulates random uniform consumption, i.e., which posits that consumers are choosing randomly uniformly from their budget frontiers. The Bronars (1987) approach has become common practice in the revealed

preference literature as a way of assessing the *power* or *precision* of revealed preference tests. We follow exactly the procedure of Choi *et al.* (2007) and generate a random sample of 25,000 simulated subjects. Each simulated subject chooses randomly uniformly from 50 budget lines that are selected in the same random fashion as in the experimental setting. The dotted curve in Figure 5 corresponds to the distribution of efficiency indices for the simulated subjects. The experimental and simulated distributions are starkly different. For example, while 80% of subjects have an efficiency index of 0.9 or higher, the chance of a randomly generated data set passing GARP at an efficiency index of 0.9 is negligible. In other words, even though the LNU model could accommodate much of the choice behavior observed in the experiment, it is also precise enough to keep out behavior that is simply randomly generated, which lends support to LNU as an accurate and discriminating model of choice among contingent consumption bundles.



(a) $\pi_1 = 1/2$ (b) $\pi \neq 1/2$

Figure 6: Distributions of efficiency indices

Going *beyond* Choi *et al.* (2007), we then calculate the distributions of efficiency indices associated with the SMU, MVU, RDU, DA, and EU models among the 93 subjects. These distributions are shown in Figures 6a and 6b, which correspond to the symmetric and asymmetric treatments, respectively. Since all of these models are more stringent than the LNU model, one would expect their efficiency indices to be lower than for the LNU model, and they are. Nonetheless, at an efficiency threshold of 0.9, around half of all subjects are consistent with the EU model, with the proportion distinctly higher under the symmetric treatment. In

the symmetric case, the performance of the EU, RDU/DA, and SMU/MVU models are very close;[26] in fact, the efficiency index distributions for the RDU/DA and SMU/MVU models are almost indistinguishable. In the asymmetric case, the distinctions between models are sharper. The RDU and SMU models appear to perform considerably better than the EU or DA models, with the distributions of their efficiency indices close to the distribution for the LNU model. Finally, the MVU model performs distinctly worse than the other models in the asymmetric case. We have not depicted any efficiency index distributions for the SMU, MVU, RDU, DA, or EU models using randomly generated data, but plainly these will be even lower than for the LNU model and therefore very different from the distributions for our experimental subjects.

$\pi_1 = 1/2$			$\pi_1 \neq 1/2$		
	$e = 0.90$	$e = 0.95$		$e = 0.90$	$e = 0.95$
LNU	38/47 (81%)	32/47 (68%)	LNU	37/46 (80%)	29/46 (63%)
SMU/MVU	30/47 (64%)	23/47 (49%)	SMU	33/46 (72%)	26/46 (57%)
			MVU	12/46 (26%)	6/46 (13%)
RDU/DA	30/47 (64%)	23/47 (49%)	RDU	33/46 (72%)	24/46 (52%)
			DA	20/46 (43%)	12/46 (26%)
EU	30/47 (64%)	18/47 (38%)	EU	18/46 (39%)	12/46 (26%)

Table 2: Pass rates by efficiency level

$\pi_1 = 1/2$			$\pi_1 \neq 1/2$		
	$e = 0.90$	$e = 0.95$		$e = 0.90$	$e = 0.95$
SMU/MVU	30/38 (79%)	23/32 (72%)	SMU	33/37 (89%)	26/29 (90%)
			MVU	12/37 (32%)	6/29 (21%)
RDU/DA	30/38 (79%)	23/32 (72%)	RDU	33/37 (89%)	24/29 (83%)
			DA	20/37 (54%)	12/29 (41%)
EU	30/38 (79%)	18/32 (56%)	EU	18/37 (49%)	12/29 (41%)

Table 3: Pass rates by efficiency level (conditional on LNU-rationalizability)

5.2 Pass rates

In order to compare the pass rates for different models more closely, we now concentrate on their performance at the 0.9 and 0.95 efficiency thresholds. These efficiency levels seem

[26] Recall that in the symmetric case, the RDU and DA models are identical and the SMU and MVU models are observationally equivalent.

like reasonable standards to set for us to consider a model to be consistent with the data; exact rationalizability is too stringent and anything less than 0.9 may be too permissive. Table 2 presents the pass rates at these thresholds. Since the different models of choice under risk which we consider are all special cases of the LNU model, it seems reasonable to focus on those subjects who are in the first place LNU-rationalizable; Table 3 presents pass rates for different models as a proportion of this group. Note that the models in Tables 2 and 3 are arranged according to their generality, with the most permissive model at the top. (The one exception is the MVU model which, under the asymmetric treatment, is more stringent than the SMU model but not comparable with the other models.) We can see that, at that 0.9 threshold, around 80% of all subjects obey the LNU model. Among this group, about half of them in turn display behavior consistent with the EU model (and, in fact, significantly more than half in the case of subjects under the symmetric treatment).

Assuming that the experimental subjects are a random sample drawn from a larger population of decision makers, we can use the sample pass rate for the EU model to estimate that model's expected population pass rate. Figure 7 depicts the 95% confidence intervals on the expected pass rates for the EU and other models. Figure 8 is similar, except that it depicts the confidence intervals on the expected pass rates conditional on LNU-rationalizability.[27]

We can see that there is some evidence that the RDU model explains a significant number of subjects not captured by the EU model. In particular, in the asymmetric case, almost 90% of subjects who are LNU-rationalizable at the 0.9 threshold are also consistent with the RDU model, which appears to be a significantly better performance than the EU model (see Table 2). Indeed, there is very little room for the RDU model to do better, since it manages to accommodate almost every subject who is consistent with the SMU model, i.e., who passes F-GARP at the same threshold. On the other hand, the DA model, which lies strictly between the RDU and EU models under the asymmetric treatment, does not perform significantly better than the EU model.[28] The MVU model, which under the asymmetric treatment is not (in theory) comparable with the EU model, clearly does not perform better than the EU model either.

[27] These intervals are calculated using the Clopper-Pearson procedure.

[28] While the contexts and methods are very different, the relatively poor performance of the DA model has been noted in some other studies, for example, Hey and Orme (1994) and Barseghyan *et al.* (2013).

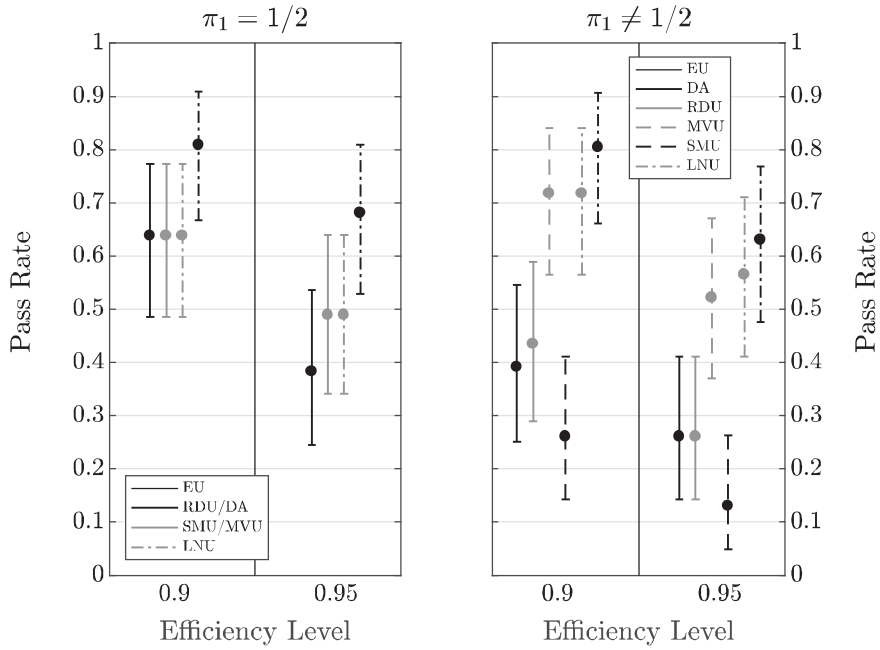


Figure 7: 95% confidence intervals on pass rates

We should be more precise on what we mean by 'significant' in the previous paragraph. We do not simply mean that the difference between the true (population) pass rates is distinct from zero; such a statistical test is not meaningful when two models are nested.[29] To go beyond this triviality, we adopt a more stringent notion of 'significant' difference. We test the null hypothesis that the difference in the expected pass rates between model A and model B *is equal to 5%*, against the alternative hypothesis that this difference is greater than 5%. In the case where model B is nested within A, we are checking whether the additional data sets which are *accommodated by model A but not B* significantly exceeds 5%. The findings of these tests are reported in Table 4. For example, at the 0.95 threshold under the symmetric treatment, Table 2 tells us that the sample proportion of subjects who *pass RDU but fail EU* is 5/47; this gives a p-value of 0.085, which is not statistically significant and so we cannot reject the null hypothesis at the 5% significance level.

We now examine more closely the performance of the RDU model under the asymmetric

[29] Suppose model A contains model B. Denoting the expected pass rates of model A (B) by μ_A (μ_B), the null hypothesis that $\mu_A = \mu_B$ is rejected if there is one data set which passes A but not B. Indeed, given that B is a special case of A, we are effectively testing the proportion of data sets which pass model A and fail B; we have to conclude that the proportion of data sets *of this type* is nonzero as long as one such data set is observed.

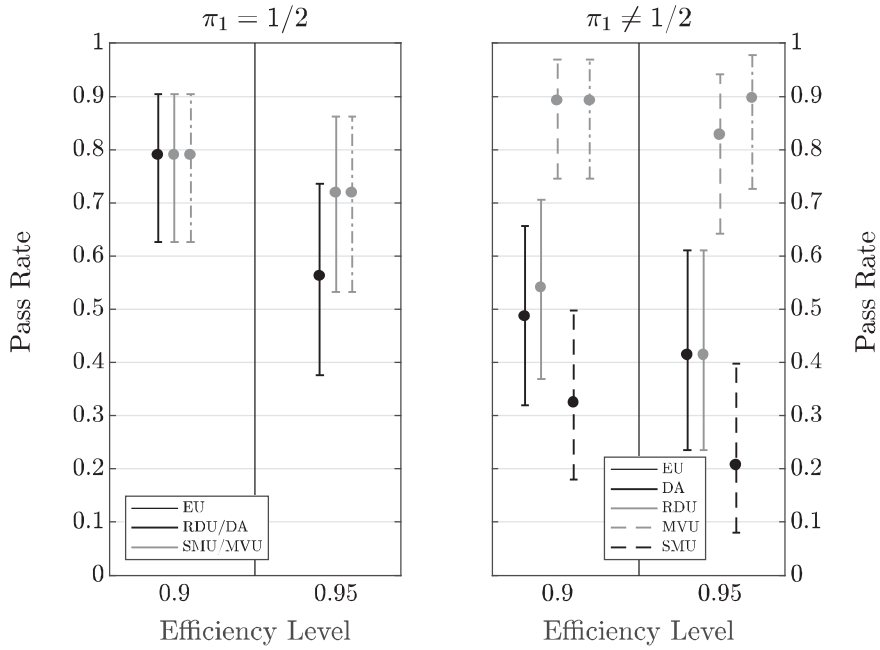


Figure 8: 95% confidence intervals on pass rates (conditional on LNU-rationalizability)

treatment. First, while the SMU and LNU models are theoretically more general, their pass rates are not significantly higher than 5% of that for the RDU model (at both threshold levels). On the other hand, the pass rate of the RDU model compared to the EU model does significantly exceed 5%. Another way of saying the same thing is that if we are to form a 90% confidence interval on the expected proportion of subjects who are RDU-rationalizable but not EU-rationalizable (when given a randomly chosen test), the lower bound of that interval exceeds 5%. What is the lower bound of that interval? At the 0.9 and 0.95 efficiency thresholds it is, respectively, 21% and 16%, which is sizeable by any reckoning.

The RDU model generalizes the EU model by permitting a distortion of the objective probabilities. When there are two states, the probability of the less favorable state is distorted to be $g(\pi)$ when π is the true probability. In the asymmetric treatment, π is either 1/3 or 2/3. It turns out that, at the 0.9 threshold, all the 15 subjects who fail EU but pass the RDU model will continue to do so if we restrict $g(2/3) \in [0.55, 0.75]$ and $g(1/3) \in [0.25, 0.45]$. (Note that g may differ across subjects.) At the 0.95 threshold, the same restrictions on the distorted probabilities will capture 11 of the 12 subjects who pass RDU (and fail EU). So it seems that those who pass the RDU test do so with fairly modest distortions of the true

$\pi_1 = 1/2$					$\pi_1 \neq 1/2$						
		LNU	SMU/MVU	RDU/DA			LNU	SMU	MVU	RDU	DA
$e = 0.95$	SMU/MVU	*0.000*	·	·	$e = 0.95$	SMU	0.406	·	·	·	·
						MVU	*0.000*	*0.000*	·	·	·
	RDU/DA	*0.000*	1.000	·		RDU	0.079	0.677	*0.000*	·	·
						DA	*0.000*	*0.000*	0.055	*0.000*	·
	EU	*0.000*	0.085	0.085		EU	*0.000*	*0.000*	0.055	*0.000*	1.000
		LNU	SMU/MVU	RDU/DA			LNU	SMU	MVU	RDU	DA
$e = 0.90$	SMU/MVU	*0.002*	·	·	$e = 0.90$	SMU	0.197	·	·	·	·
						MVU	*0.000*	*0.000*	·	·	·
	RDU/DA	*0.002*	1.000	·		RDU	0.197	1.000	*0.000*	·	·
						DA	*0.000*	*0.000*	*0.027*	*0.000*	·
	EU	*0.002*	1.000	1.000		EU	*0.000*	*0.000*	0.087	*0.000*	0.677

Note: Each cell contains a p-value, with values below 0.05 italicized.

Table 4: Pairwise 5%-differences in pass rates

probabilities.[30]

Furthermore, there is some evidence that subjects deflate the probability of the less favorable state when it is objectively 2/3 and inflate the probability when it is 1/3, so that the cumulative probability weighting function has the shape favored by cumulative prospect theory. Indeed, if we restrict ourselves to choosing $g(2/3) \in [0.55, 2/3]$ and $g(1/3) \in [1/3, 0.45]$, we still capture every subject who passes the RDU model at the 0.9 threshold and all but two of the subjects who pass RDU at the 0.95 threshold. On the other hand, the mirror restriction performs very badly: if we insist on choosing $g(2/3) \in [2/3, 0.75]$ and $g(1/3) \in [0.25, 1/3]$, the RDU model will capture *no subject* at both efficiency thresholds other than those who are EU-rationalizable.[31]

The results reported in Table 4 support the informal observations we have already made on the relative performance of different models: (1) the LNU model has significantly higher pass rates, compared to all other models under the symmetric treatment and compared to the MVU, DA, and EU models under asymmetric treatment; (2) in the symmetric treatment, the performance of the SMU/MVU, RDU/DA, and EU models are not significantly different

[30] We know from Table 2 that, for the symmetric treatment the pass rates of the EU and DA/RDU models differ only at the efficiency threshold 0.95, where 5 subjects pass the DA/RDU model but fail EU. All 5 subjects will pass the RDU test for some $g(1/2) < 0.5$ (which is consistent with disappointment aversion) and 4 of them will pass with values of $g(1/2)$ chosen from the interval $[0.45, 0.5)$.

[31] Note that for any subject who passes RDU, there will typically be more than one set of distorted probabilities at which the subject will be consistent with the model.

from one another; (3) in the asymmetric treatment, SMU and RDU perform significantly better than the EU model, while the DA and MVU models do not.

5.3 Power

We learned in Section 5.2 that some models have significantly higher pass rates than other models. In general, we would view a model with a higher pass rate more favorably, but it is clear that a model's value also depends on the sharpness of its predictions: a model which excludes no outcome will have a perfect pass rate but such a model is clearly useless. We now examine more closely the power of different models, adopting and then adapting the approach first suggested by Bronars (1987).

Bronars (1987) calculated the power of the LNU model by generating data sets where on each budget set, the bundle is randomly chosen based on a uniform distribution on the budget plane. The *power* of the model (or its *Bronars power*) is then given by the probability of such a data set being inconsistent with the LNU model, which is synonymous with it failing GARP. As we have already pointed out in Section 5.2, when the data set consists of bundles on 50 randomly chosen budget sets, the Bronars power is very strong. To be specific, both at an efficiency threshold of 0.95 and also at 0.9, the power is approximately 1; in other words, the probability of such a randomly generated data set passing GARP at those thresholds is vanishingly small. Obviously, the Bronars power of the other models we are considering is also roughly equal to 1, since all of them imply locally nonsatiated preferences.

$\pi_1 = 1/2$			$\pi_1 \neq 1/2$		
	$e = 0.90$	$e = 0.95$		$e = 0.90$	$e = 0.95$
SMU/MVU	1.00	1.00	SMU	0.88	0.92
			MVU	1.00	1.00
RDU/DA	1.00	1.00	RDU	1.00	1.00
			DA	1.00	1.00
EU	1.00	1.00	EU	1.00	1.00

Table 5: Power (conditional on LNU-rationalizability)

When we are considering a model that we know is theoretically more stringent than the LNU model, it is natural to investigate the power of the model *in the context of observed behavior that is consistent with the LNU model.* In other words, we would like to know

the sharpness of the model's predictions relative to the LNU model. For example, to check the relative power of the EU model in this sense, we randomly generate data sets that are consistent with the LNU model at a given threshold,[32] and then test if they obey the EU model at the same threshold; when the sample is sufficiently large,[33] we obtain an accurate estimate of the power of the EU model relative to the LNU model. With the exception of the LNU model, all of the utility models we consider are also consistent with stochastic monotonicity, so it is also natural to investigate the power of, say, the EU or RDU model, *relative to the SMU model*; this would give us a sense of how stringent are the restrictions imposed by the EU or RDU model, over and above those imposed by stochastic monotonicity. To do this, we randomly generate a large collection of data sets consistent with the SMU model at a given efficiency threshold[34] and then check whether they pass the EU or RDU models at the same threshold.

Table 5 presents the power of the different models, conditional on LNU-rationalizability. The most obvious and important feature in this table is the ubiquity of the number 1: even after conditioning on LNU-rationalizability, all of the models remain very precise. For example, the probability of an LNU-rationalizable data set passing the EU test at the 0.9 or 0.95 threshold is effectively zero. The only partial exception to this rule is the SMU model, where the power is around 90%. A reader who has the intuition that stochastic monotonicity leads to sharp restrictions on data over and above those required by the maximization of a

[32] The process of generating a random data set obeying GARP at a given efficiency threshold is as follows. First, we generate 50 budget sets according to the procedure specified in Choi *et al.* We randomly select a budget line and then randomly choose a bundle on that line. We next randomly select a second budget line and then randomly choose a bundle from that part of the line which guarantees that this observation, along with the first, obeys GARP at the given efficiency threshold. A third budget line is then randomly selected and a bundle randomly chosen from that part of its budget line so that all three observations together obey GARP. Note that there must exist such a bundle on the third budget line; for example, the demand (on the third budget line) arising from any utility function rationalizing the first two observations will have this property. We then choose a fourth budget line randomly, a bundle on that line so that the first four observations obey GARP, and so on.

[33] By the Azuma-Hoeffding inequality, in order to be $100(1-\delta)$ percent confident that the sample pass rate resulting from a simulation is within ϵ of the true probability of passing the test, we require at least $N = (1/2\epsilon^2)\log(2/\delta)$ samples. With 30,000 samples, we can be 99.5 percent sure that our estimate is within 0.01 of the true value.

[34] For the reason given in footnote 33, we generate 30,000 data sets that are consistent with the SMU model at each of the two efficiency thresholds. The method used to generate such data sets is analogous to the procedure laid out in footnote 32 for constructing a random LNU-consistent data set, except that F-GARP is used in place of GARP.

locally nonsatiated utility function may not be overly surprised by this finding, so let us go one step further and investigate what happens when we condition on SMU-rationalizability.

$\pi_1 = 1/2$			$\pi_1 \neq 1/2$		
	$e = 0.90$	$e = 0.95$		$e = 0.90$	$e = 0.95$
MVU	0.00	0.00	MVU	1.00	1.00
RDU/DA	0.75	0.99	RDU	0.99	1.00
			DA	1.00	1.00
EU	0.87	0.99	EU	1.00	1.00

Table 6: Power (conditional on SMU-rationalizability)

For the asymmetric case, we see from Table 6 that all the models are very precise, even in the context of data sets consistent with the SMU model. In particular, while models like DA and RDU are more permissive than the EU model, they still deliver far sharper predictions than those imposed merely by a stochastically monotone utility function. In the symmetric case, the MVU model has no power at all, since it is observationally equivalent to the SMU model. For the other three models, the relative power is now noticeably lower than 1; for example, a quarter of all subjects who are consistent with the SMU model at the 0.9 threshold are also consistent with the RDU model. A possible reason for the loss of relative power in the symmetric case is that the SMU model itself is very restrictive in this context since it is synonymous with an increasing *and symmetric* utility function. Having said that, we should also note that the relative power of the RDU, DA, and EU models remain high.

5.4 Predictive success

So far we have separately considered the pass rates and power of different models of choice under risk. Clearly, a comparison of pass rates will tend to favor the more inclusive models, while a comparison of power will tend to favor theoretically more stringent models such as EU. In this section, we combine these two criteria into a single measure: the *index of predictive success* proposed by Selten (1991), which we shall refer to simply as the *Selten index*. This index is defined as the difference between the frequency of correct predictions (the pass rate) and the size of the set of predicted outcomes (the imprecision), with the latter typically measured by a uniform measure on all outcomes. Selten provides an axiomatic

foundation for this index. Note that the value of the index varies between 1 and -1. An index that is close to 1 occurs when the pass rate is close to 1 and the imprecision is close to zero; in other words, even though the model is very precise in its predictions (i.e., has a high power), the data collected are very often consistent with the model. On the other hand, an index of -1 occurs when the hit rate is close to zero even though the model is very imprecise (i.e., has a low power). An index above zero indicates that the model has some predictive success. Our use of the Selten index to evaluate different models is not novel. In the context of consumer demand (which formally is very similar to ours), it has been used by Beatty and Crawford (2011); it has also been used by Harless and Camerer (1994) to compare the performance of different models of choice under risk.

As we have emphasized, when there are 50 observations per subject (as we do in the data collected by Choi *et al.*), the LNU model has a power that is almost indistinguishable from 1 at the 0.9 and 0.95 efficiency thresholds. The same is true of course of all the other models, since they are more restrictive than the LNU model. In other words, all the models have an imprecision of zero, so that the Selten indices for the different models are effectively given by their pass rates, as displayed in Table 2 . Note also that because the different models are nested within one another, any observed differences (from zero) in the pass rates/Selten indices in Table 2 are all statistically significant.[35] One conclusion to be drawn from this table is that the best model, as evaluated by the Selten index, is the LNU model. This observation may be simple but it is not without interest: while a great deal of academic discussion is often focussed on comparing different models that have been tailor-made for decision making under risk, we should *not* take it for granted that such models are necessarily better than basic LNU in modelling choice behavior under risk. In environments where state payoffs vary while state probabilities are fixed, such as in the experimental choices analyzed here, one should not exclude the possibility that the LNU model does a better job in explaining the data, even after accounting for its relative lack of specificity.[36]

[35] The one exception is the MVU model in the asymmetric case which is not nested with the EU, DA, or RDU models. Nonetheless, the observed differences in the pass rates/Selten indices between MVU and the EU, DA, and RDU models are statistically significant. At the 0.95 efficiency threshold, a two-tailed test of the hypothesis that MVU has the same pass rate/Selten index as RDU, DA, and EU give p-values of 0.000, 0.009, and 0.009 respectively. At the 0.9 threshold, the corresponding p-values are 0.000, 0.007, and 0.0027.

[36] This observation is not an argument against the value of models of choice under risk, such as SMU, RDU, and all the other models considered here. In particular, these other models provide a theory of choice

$\pi_1 = 1/2$			$\pi_1 \neq 1/2$		
	$e = 0.90$	$e = 0.95$		$e = 0.90$	$e = 0.95$
SMU/MVU	0.79	0.72	SMU	0.77	0.81
			MVU	0.32	0.21
RDU/DA	0.79	0.72	RDU	0.89	0.83
			DA	0.54	0.41
EU	0.79	0.56	EU	0.49	0.41

Table 7: Predictive success (conditional on LNU-rationalizability)

Our next objective is to investigate *conditional predictive success*. By this we mean the predictive success of a model after we restrict attention to LNU- or SMU-rationalizable behavior.

We first turn to the case where we condition on LNU-rationalizability. The conditional pass rate of each model is given in Table 3. A model's imprecision is simply 1 minus the Bronars power (conditional on LNU-rationalizability) and this is supplied in Table 5. The Selten indices can be easily constructed by taking the difference between the conditional pass rate and the conditional imprecision. The results are displayed in Table 7. For the symmetric treatment, all the models have a power of approximately 1 (see Table 5), so the Selten indices are completely determined by the conditional pass rates and, as such, the differences between the SMU/MVU, RDU/DA, and EU models are not large. For the asymmetric treatment, it is notable that the best performing model is RDU; this is driven by two factors: its pass rate is higher than all models except SMU and its power is higher than SMU. A statistical analysis of the differences in the Selten indices is provided in Table 8. Each entry in the table gives the p-value of the test of null hypothesis that the two models have the same Selten index, with the alternative hypothesis that they do not.[37] The differences in Selten

across *all* lotteries, allowing even for comparisons between lotteries where the same outcomes occur with different probabilities. In environments where agents are making choices among lotteries of this type, all the other models are still potentially applicable, but it is not clear how one could naturally generalize the LNU model to capture such a context.

[37] To illustrate how this test is carried out, consider a test of the equality of the Selten indices between the EU and SMU models, under the asymmetric treatment and at the efficiency threshold 0.9. As shown in Table 5, the SMU model has a power of 0.88 and the EU model has a power of 1; we take this as given. The null hypothesis that the Selten indices are equal is equivalent to the hypothesis that $\mu_{SMU} - \mu_{EU} = 1 - 0.88 = 0.12$, where μ_{EU} and μ_{SMU} are the expected pass rates for the EU and SMU models (in the population of LNU-rationalizable data sets). The alternative hypothesis is $\mu_{SMU} - \mu_{EU} \neq 0.12$. The sample estimate of $\mu_{SMU} - \mu_{EU}$ has a binomial distribution and its realized value is $(33 - 18)/37$; according to Table 8, the probability of obtaining this sample estimate or something more extreme if $\mu_{SMU} - \mu_{EU} = 0.12$ is effectively

$\pi_1 = 1/2$				$\pi_1 \neq 1/2$					
$e = 0.95$		SMU/MVU	RDU/DA	$e = 0.95$		SMU	MVU	RDU	DA
					MVU	*0.000*	·	·	·
	RDU/DA	1.000	·		RDU	0.432	*0.000*	·	·
					DA	*0.000*	*0.009*	*0.000*	·
	EU	*0.000*	*0.000*		EU	*0.000*	*0.009*	*0.000*	1.000
$e = 0.90$		SMU/MVU	RDU/DA	$e = 0.90$		SMU	MVU	RDU	DA
					MVU	*0.000*	·	·	·
	RDU/DA	1.000	·		RDU	*0.005*	*0.000*	·	·
					DA	*0.002*	*0.007*	*0.000*	·
	EU	1.000	1.000		EU	*0.000*	*0.027*	*0.000*	*0.000*

Note: Each cell contains a p-value, with values below 0.05 italicized.

Table 8: Pairwise differences in predictive success
(conditional on LNU-rationalizability)

$\pi_1 = 1/2$			$\pi_1 \neq 1/2$		
	$e = 0.90$	$e = 0.95$		$e = 0.90$	$e = 0.95$
MVU	0.00	0.00	MVU	0.36	0.23
RDU/DA	0.75	0.99	RDU	0.99	0.92
			DA	0.61	0.46
EU	0.87	0.78	EU	0.55	0.46

Table 9: Predictive success (conditional on SMU-rationalizability)

indices between the RDU and EU, DA, and MVU models are all statistically significant. It is also worth remarking that the performance of the MVU model under the asymmetric treatment is the worst of all the models, principally because its pass rate is the lowest, and this difference is statistically significant.

Lastly, we investigate the predictive success of the EU, DA, and RDU models when we condition on SMU-rationalizable behavior. The Selten indices displayed in Table 9 are obtained by taking the difference between the sample pass rates (constructed from Table 2) and the power (from Table 6). Focussing firstly on the symmetric treatment, an interesting phenomenon is that, according to the Selten index, the EU model is now better than the RDU/DA model at the 0.9 efficiency threshold; this is entirely driven by the greater power of the EU model in this context. That said, the difference between the indices is not large and it is also reversed at the 0.95 threshold. For the asymmetric treatment, we notice that the RDU

zero.

$\pi_1 = 1/2$				$\pi_1 \neq 1/2$				
$e = 0.95$		MVU	RDU/DA	$e = 0.95$		MVU	RDU	DA
	RDU/DA	*0.000*	·		RDU	*0.000*	·	·
					DA	*0.009*	*0.000*	·
	EU	*0.000*	*0.000*		EU	*0.009*	*0.000*	1.000
$e = 0.90$		MVU	RDU/DA	$e = 0.90$		MVU	RDU	DA
	RDU/DA	*0.000*	·		RDU	*0.000*	·	·
					DA	*0.007*	*0.000*	·
	EU	*0.000*	*0.003*		EU	*0.027*	*0.000*	*0.000*

Note: Each cell contains a p-value, with values below 0.05 italicized.

Table 10: Pairwise differences in predictive success
(conditional on SMU-rationalizability)

model performs well relative to the other models, because its pass rate is high *and* because the model continues to have high power, even after conditioning on SMU-rationalizability; the p-values reported in Table 10 confirm that the differences are statistically significant.

6. Conclusion

In this paper, we have demonstrated how recent advances in revealed preference techniques can be employed to evaluate the performance of different models of choice under risk and under uncertainty in the context of data sets containing budgetary observations. A fairly conspicuous limitation of our empirical conclusions is that it is based on experimental data where, for a given subject, the state probabilities are fixed across all observations. It would be interesting to see how our conclusions would change if (say) a subject is observed to choose from different budget sets, where state probabilities may vary across observations.[38] Note that our basic methodological result, Theorem 1, allows for this possibility since it allows the utility function to change with the observation (as it will if state probabilities are allowed to vary), but our empirical implementation does not present an occasion for us to exploit this feature.

To conclude, we believe that there is a great deal of worthwhile empirical work that can be pursued using and extending the ideas and techniques developed in this paper, on richer or different data sets. This future work could add to, support, or perhaps even challenge the

[38] In such a context it still makes sense to check consistency with the SMU, RDU, DA, EU, and MVU models, but it is not clear how one could naturally generalize LNU-rationalizability. See also footnote 36.

various empirical findings we have obtained.

Appendix

The proof of Theorem 1 uses the following lemma.

Lemma 1. *Let $\{C^t\}_{t=1}^T$ be a finite collection of constraint sets in $\mathbb{R}_+^{\bar{s}}$ that are compact and downward closed (i.e., if $\mathbf{x} \in C^t$ then so is $\mathbf{y} \in \mathbb{R}_+^{\bar{s}}$ such that $\mathbf{y} < \mathbf{x}$) and let the functions $\{\phi(\cdot, t)\}_{t=1}^T$ be continuous and increasing in all dimensions. Suppose that there is a finite set $\mathcal{X}$ of $\mathbb{R}_+$, a strictly increasing function $\bar{u} : \mathcal{X} \to \mathbb{R}_+$, and $\{M^t\}_{t=1}^T$ such that the following holds:*

$$M^t \geqslant \phi(\bar{\mathbf{u}}(\mathbf{x}), t) \text{ for all } \mathbf{x} \in C^t \cap \mathcal{L} \text{ and} \tag{17}$$

$$M^t > \phi(\bar{\mathbf{u}}(\mathbf{x}), t) \text{ for all } \mathbf{x} \in (C^t \backslash \partial C^t) \cap \mathcal{L}, \tag{18}$$

where $\mathcal{L} = \mathcal{X}^{\bar{s}}$ and $\bar{\mathbf{u}}(\mathbf{x}) = (\bar{u}(x_1), \bar{u}(x_2), \ldots, \bar{u}(x_{\bar{s}}))$. Then there is a Bernoulli function $u : \mathbb{R}_+ \to \mathbb{R}_+$ that extends $\bar{u}$ such that

$$M^t \geqslant \phi(\mathbf{u}(\mathbf{x}), t) \text{ for all } \mathbf{x} \in C^t \text{ and} \tag{19}$$

$$\text{if } \mathbf{x} \in C^t \text{ and } M^t = \phi(\mathbf{u}(\mathbf{x}), t), \text{ then } \mathbf{x} \in \partial C^t \cap \mathcal{L} \text{ and } M^t = \phi(\bar{\mathbf{u}}(\mathbf{x}), t). \tag{20}$$

Remark: The property (20) needs some explanation. Conditions (17) and (18) allow for the possibility that $M^t = \phi(\bar{\mathbf{u}}(\mathbf{x}'), t)$ for some $\mathbf{x}' \in \partial C^t \cap \mathcal{L}$; we denote the set of points in $\partial C^t \cap \mathcal{L}$ with this property by X'. Clearly any extension u will preserve this property, i.e., $M^t = \phi(\mathbf{u}(\mathbf{x}'), t)$ for all $\mathbf{x}' \in X'$. Property (20) says that we can choose u such that for all $\mathbf{x} \in C^t \backslash X'$, we have $M^t > \phi(\mathbf{u}(\mathbf{x}), t)$.

Proof: We shall prove this result by induction on the dimension of the space containing the constraint sets. It is trivial to check that the claim is true if $\bar{s} = 1$. In this case, $\mathcal{L}$ consists of a finite set of points on $\mathbb{R}_+$ and each C^t is a closed interval with 0 as its minimum. Now let us suppose that the claim holds for $\bar{s} = m$ and we shall prove it for $\bar{s} = m + 1$. If, for each t, there is a strictly increasing and continuous utility function $u^t : \mathbb{R}_+ \to \mathbb{R}_+$ extending $\bar{u}$ such that (19) and (20) hold, then the the same conditions will hold for the increasing and continuous function $u = \min_t u^t$. So we can focus our attention on constructing u^t for a single constraint set C^t.

Suppose $\mathcal{X} = \{0, r^1, r^2, r^3, \ldots, r^I\}$, with $r^0 = 0 < r^i < r^{i+1}$, for $i = 1, 2, \ldots, I-1$. Let $\bar{r} = \max\{r \in \mathbb{R}_+ : (r, 0, 0, \ldots, 0) \in C^t\}$ and suppose that $(r^i, 0, 0, \ldots, 0) \in C^t$ if and only if $i \leqslant N$ (for some $N \leqslant I$). Consider the collection of sets of the form $D^i = \{\mathbf{y} \in \mathbb{R}^m_+ : (r^i, \mathbf{y}) \in C^t\}$ (for $i = 1, 2, \ldots, N$); this is a finite collection of compact and downward closed sets in $\mathbb{R}^m_+$. By the induction hypothesis applied to $\{D^i\}_{i=1}^N$, with $\{\phi(\bar{u}(r^i), \cdot, t)\}_{i=1}^N$ as the collection of functions, there is a strictly increasing function $u^* : \mathbb{R}_+ \to \mathbb{R}_+$ extending $\bar{u}$ such that

$$M^t \geqslant \phi(\bar{u}(r^i), \mathbf{u}^*(\mathbf{y}), t) \text{ for all } (r^i, \mathbf{y}) \in C^t \text{ and} \tag{21}$$

$$\text{if } (r^i, \mathbf{y}) \in C^t \text{ and } M^t = \phi(\bar{u}(r^i), \mathbf{u}^*(\mathbf{y}), t), \text{ then } (r^i, \mathbf{y}) \in \partial C^t \cap \mathcal{L} \text{ and } M^t = \phi(\bar{\mathbf{u}}(r^i, \mathbf{y}), t). \tag{22}$$

For each $r \in [0, \bar{r}]$, define

$$U(r) = \{u \leqslant u^*(r) : \max\{\phi(u, \mathbf{u}^*(\mathbf{y}), t) : (r, \mathbf{y}) \in C^t\} \leqslant M^t\}.$$

This set is nonempty; indeed $\bar{u}(r^k) = u^*(r^k) \in U(r)$, where r^k is the largest element in $\mathcal{X}$ that is weakly smaller than r. This is because, if $(r, \mathbf{y}) \in C^t$ then so is $(r^k, \mathbf{y})$, and (21) guarantees that $\phi(\bar{u}(r^k), \mathbf{u}^*(\mathbf{y}), t) \leqslant M^t$. The downward closedness of C^t and the fact that u^* is increasing also guarantees that $U(r) \subseteq U(r')$ whenever $r < r'$. Now define $\tilde{u}(r) = \sup U(r)$; the function $\tilde{u}$ has a number of significant properties. (i) *For $r \in \mathcal{X}$, $\tilde{u}(r) = u^*(r) = \bar{u}(r)$* (by the induction hypothesis). (ii) *$\tilde{u}$ is a nondecreasing function* since U is nondecreasing. (iii) *$\tilde{u}(r) > \bar{u}(r^k)$ if $r > r^k$, where r^k is largest element in $\mathcal{X}$ smaller than r.* Indeed, because C^t is compact and ϕ continuous, $\phi(\tilde{u}(r), \mathbf{u}^*(\mathbf{y}), t) \leqslant M^t$ for all $(r, \mathbf{y}) \in C^t$. By way of contradiction, suppose $\tilde{u}(r) = \bar{u}(r^k)$ and hence $\tilde{u}(r) < u^*(r)$. It follows from the definition of $\tilde{u}(r)$ that, for any sequence u_n, with $\tilde{u}(r) < u_n < u^*(r)$ and $\lim_{n\to\infty} u_n = \tilde{u}(r)$, there is $(r, \mathbf{y}_n) \in C^t$ such that $\phi(u_n, \mathbf{u}^*(\mathbf{y}_n), t) > M^t$. Since C^t is compact, we may assume with no loss of generality that $\mathbf{y}_n \to \hat{\mathbf{y}}$ and $(r, \hat{\mathbf{y}}) \in C^t$, from which we obtain $\phi(\tilde{u}(r), \mathbf{u}^*(\hat{\mathbf{y}}), t) = M^t$. Since C^t is downward closed, $(r^k, \hat{\mathbf{y}}) \in C^t$ and, since $\bar{u}(r^k) = u^*(r^k)$, we have $\phi(\mathbf{u}^*(r^k, \hat{\mathbf{y}}), t) = M^t$. This can only occur if $(r^k, \hat{\mathbf{y}}) \in \partial C^t \cap \mathcal{L}$ (because of (22)), but it is clear that $(r^k, \hat{\mathbf{y}}) \notin \partial C^t$ since $(r^k, \hat{\mathbf{y}}) < (r, \hat{\mathbf{y}})$. (iv) *If $r_n < r^i$ for all n and $r_n \to r^i \in \mathcal{X}$, then $\tilde{u}(r_n) \to u^*(r^i)$.* Suppose to the contrary, that the limit is $\hat{u} < u^*(r^i) = \bar{u}(r^i)$. Since u^* is continuous, we can assume, without loss of generality, that $\tilde{u}(r_n) < u^*(r_n)$. By the compactness of C^t, the continuity of ϕ, and the definition of $\tilde{u}$, there is $(r_n, \mathbf{y}_n) \in C^t$ such that $\phi(\tilde{u}(r_n), \mathbf{u}^*(\mathbf{y}_n), t) = M^t$. This

leads to $\phi(\hat{u}, \mathbf{u}^*(\mathbf{y}'), t) = M^t$, where $\mathbf{y}'$ is an accumulation point of $\mathbf{y}_n$ and $(r^i, \mathbf{y}') \in C^t$. But since ϕ is strictly increasing, we obtain $\phi(u^*(r^i), \mathbf{u}^*(\mathbf{y}'), t) > M^t$, which contradicts (21).

Given the properties of $\tilde{u}$, we can find a continuous and strictly increasing function u^t such that u^t extends $\bar{u}$, i.e., $u^t(r) = \bar{u}(r)$ for $r \in \mathcal{X}$, $u^t(r) < u^*(r)$ for all $r \in \mathbb{R}_+ \backslash \mathcal{X}$ and $u^t(r) < \tilde{u}(r) \leqslant u^*(r)$ for all $r \in [0, \bar{r}] \backslash \mathcal{X}$. (In fact we can choose u^t to be smooth everywhere except possibly on $\mathcal{X}$.) We claim that (19) and (20) are satisfied for C^t. To see this, note that for $r \in \mathcal{X}$ and $(r, \mathbf{y}) \in C^t$, the induction hypothesis guarantees that (21) and (22) hold and they will continue to hold if u^* is replaced by u^t. In the case where $r \notin \mathcal{X}$ and $(r, \mathbf{y}) \in C^t$, since $u^t(r) < \tilde{u}(r)$ and ϕ is increasing, we obtain $M^t > \phi(\mathbf{u}^t(r, \mathbf{y}), t)$. **QED**

Proof of Theorem 1: This follows immediately from Lemma 1 if we set $C^t = \underline{B}^t$, and $M^t = \phi(\bar{\mathbf{u}}(\mathbf{x}^t), t)$. If $\bar{u}$ obeys conditions (7) and (8) then it obeys conditions (17) and (18). The rationalizability of $\mathcal{O}$ by $\{\phi(\cdot, t)\}_{t \in T}$ then follows from (19). **QED**

Description of the RDU-rationalizability test for multiple states: Suppose that $\pi_s > 0$ is the objective probability of state s. To develop a necessary and sufficient test for RDU-rationalizability, we first define $\Gamma = \{\sum_{s \in S} \pi_s : S \subseteq \{1, 2, \ldots, \bar{s}\}\}$, i.e., Γ is a finite subset of $[0, 1]$ that includes both 0 and 1 (corresponding to S equal to the empty set and the whole set, respectively). Suppose *there are strictly increasing functions* $\bar{g} : \Gamma \to \mathbb{R}$ *and* $\bar{u} : \mathcal{X} \to \mathbb{R}_+$ *such that (7) and (8) are satisfied, with* $\phi(\mathbf{u}) = \sum_{s=1}^{\bar{s}} \delta(\mathbf{u}, s) u_s$ *and*

$$\delta(\mathbf{u}, s) = \bar{g}\left(\sum_{\{s' : r(\mathbf{u}, s') \leqslant r(\mathbf{u}, s)\}} \pi_{s'}\right) - \bar{g}\left(\sum_{\{s' : r(\mathbf{u}, s') < r(\mathbf{u}, s)\}} \pi_{s'}\right).$$

By Theorem 1, this guarantees that $\mathcal{O}$ is RDU-rationalizable, with $g : [0, 1] \to \mathbb{R}$ chosen to be any strictly increasing extension of $\bar{g}$. This test involves finding a solution to a set of inequalities that are bilinear in the unknowns $\{\bar{g}(\gamma)\}_{\gamma \in \Gamma}$ and $\{\bar{u}(r)\}_{r \in \mathcal{X}}$. It is also clear that these conditions are necessary for RDU-rationalizability since they will be satisfied if we simply let $\bar{g}$ and $\bar{u}$ be the restrictions of g and u to Γ and $\mathcal{X}$ respectively. **QED**

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Online Appendix

to

Revealed preferences over risk and uncertainty

Matthew Polisson, John K.-H. Quah, and Ludovic Renou

A1. Introduction

This Online Appendix consists of two parts. In Section A.2, we discuss further applications of the lattice method which were not covered in the main text. In particular, we cover the choice acclimating personal equilibrium (CPE) model (Köszegi and Rabin, 2007), the maxmin expected utility (MEU) model (Gilboa and Schmeidler, 1989), the variational preference (VP) model (Maccheroni, Marinacci, and Rustichini, 2006), and a model with budget-dependent reference points. In Section A.3, we discuss the relationship between the stochastically monotone utility (SMU) and mean-variance utility (MVU) models in a choice environment with only two states and budgetary observations, confirming the claims made in Section 4 of the main paper.

A2. Further applications of the lattice method

Theorem 1 in the main text can also be used to test the rationalizability of many other models of choice under risk and under uncertainty. Formally, this involves finding a Bernoulli function u and a function ϕ belonging to some family Φ (corresponding to the particular model at hand) which together rationalize the data. In the subjective expected utility (SEU) case that we discussed in the main text, we know that the lattice test involves solving a system of inequalities that are bilinear in the utility levels $\{\bar{u}(r)\}_{r\in\mathcal{X}}$ and the subjective probabilities $\{\pi_s\}_{s=1}^{\bar{s}}$. Such a formulation seems natural enough in the SEU case; what is

worth remarking (and perhaps not obvious *a priori*) is that the *same* pattern holds across many of the common models of choice under risk and under uncertainty: they can be tested by solving a system of inequalities that are bilinear in $\{\bar{u}(r)\}_{r\in\mathcal{X}}$ and a finite set of variables specific to the particular model in question. It is known that bilinear systems are decidable, in the sense that there is an algorithm that can determine in a finite number of steps whether or not a solution exists. In addition to the expected utility (EU), disappointment aversion (DA), and rank dependent utility (RDU) models which we covered in the main text, in this section we further illustrate the flexibility of the lattice method by applying it to several prominent models of decision making under risk or uncertainty.

A2.1 Choice acclimating personal equilibrium

The choice acclimating personal equilibrium (CPE) model (Köszegi and Rabin, 2007) (with a piecewise linear gain-loss function) specifies utility as $V(\mathbf{x}) = \phi(\mathbf{u}(\mathbf{x}), \boldsymbol{\pi})$, where

$$\phi((u_1, u_2, \ldots, u_{\bar{s}}), \boldsymbol{\pi}) = \sum_{s=1}^{\bar{s}} \pi_s u_s + \frac{1}{2}(1-\lambda) \sum_{r,s=1}^{\bar{s}} \pi_r \pi_s |u_r - u_s|, \tag{A.1}$$

$\boldsymbol{\pi} = \{\pi_s\}_{s=1}^{\bar{s}}$ are the objective probabilities, and $\lambda \in [0,2]$ is the coefficient of loss aversion.[1] We say that a data set $\mathcal{O} = \{(\mathbf{x}^t, B^t)\}_{t=1}^T$ is CPE-rationalizable with the probability weights $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_{\bar{s}}) \gg 0$ if there is ϕ in the collection Φ_{CPE} of functions of the form (A.1), and a Bernoulli function $u : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for each t, $\phi(\mathbf{u}(\mathbf{x}^t), \boldsymbol{\pi}^t) \geqslant \phi(\mathbf{u}(\mathbf{x}), \boldsymbol{\pi}^t)$ for all $\mathbf{x} \in B^t$. Applying Theorem 1 in the main text, *$\mathcal{O}$ is CPE-rationalizable if and only if there is $\lambda \in [0,2]$ and a strictly increasing function $\bar{u} : \mathcal{X} \to \mathbb{R}_+$ that solve (7) and (8) in the main text.* It is notable that, irrespective of the number of states, this test is linear in the remaining variables for any given value of λ. Thus it is relatively straightforward to implement via a collection of linear tests (running over different values of $\lambda \in [0,2]$).

A2.2 Maxmin expected utility

We again consider a setting where no objective probabilities can be attached to each state. An agent with maxmin expected utility (MEU), first presented by Gilboa and Schmeidler

[1] Our presentation of CPE follows Masatlioglu and Raymond (2016). The restriction of λ to $[0,2]$ guarantees that V respects first order stochastic dominance but allows for loss-loving behavior (see Masatlioglu and Raymond (2016)).

(1989), evaluates each bundle $\mathbf{x} \in \mathbb{R}^{\bar{s}}_{+}$ using the formula $V(\mathbf{x}) = \phi(\mathbf{u}(\mathbf{x}))$, where

$$\phi(\mathbf{u}) = \min_{\boldsymbol{\pi} \in \Pi} \left\{ \sum_{s=1}^{\bar{s}} \pi_s u_s \right\}, \tag{A.2}$$

where $\Pi \subset \Delta_{++} = \{\boldsymbol{\pi} \in \mathbb{R}^{\bar{s}}_{++} : \sum_{s=1}^{\bar{s}} \pi_s = 1\}$ is nonempty, compact in $\mathbb{R}^{\bar{s}}$, and convex. (Π can be interpreted as a set of probability weights.) Given these restrictions on Π, the minimization problem in (A.2) always has a solution and ϕ is strictly increasing.

A data set $\mathcal{O} = \{(\mathbf{x}^t, B^t)\}_{t=1}^{T}$ is said to be MEU-rationalizable if there is a function ϕ in the collection Φ_{MEU} of functions of the form (A.2), and a Bernoulli function $u : \mathbb{R}_{+} \to \mathbb{R}_{+}$ such that, for each t, $\phi(\mathbf{u}(\mathbf{x}^t), \boldsymbol{\pi}^t) \geqslant \phi(\mathbf{u}(\mathbf{x}), \boldsymbol{\pi}^t)$ for all $\mathbf{x} \in B^t$. By Theorem 1 in the main text, this holds if and only if there exist Π and $\bar{u}$ that solve (7), (8), and (9) in the main text. We claim that this requirement can be reformulated in terms of the solvability of a set of bilinear inequalities.

This is easy to see for the two-state case where we may assume, without loss of generality, that there is π_1^* and $\pi_1^{**} \in (0,1)$ such that $\Pi = \{(\pi_1, 1-\pi_1) : \pi_1^* \leqslant \pi_1 \leqslant \pi_1^{**}\}$. Then it is clear that $\phi(u_1, u_2) = \pi_1^* u_1 + (1-\pi_1^*)u_2$ if $u_1 \geqslant u_2$ and $\phi(u_1, u_2) = \pi_1^{**} u_1 + (1-\pi_1^{**})u_2$ if $u_1 < u_2$. Consequently, for any $(x_1, x_2) \in \mathcal{L}$, we have $V(x_1, x_2) = \pi_1^* \bar{u}(x_1) + (1-\pi_1^*)\bar{u}(x_2)$ if $x_1 \geqslant x_2$ and $V(x_1, x_2) = \pi_1^{**}\bar{u}(x_1) + (1-\pi_1^{**})\bar{u}(x_2)$ if $x_1 < x_2$ and this is independent of the precise choice of $\bar{u}$. Therefore, *$\mathcal{O}$ is MEU-rationalizable if and only if we can find π_1^* and π_1^{**} in $(0,1)$, with $\pi_1^* \leqslant \pi_1^{**}$, and an increasing function $\bar{u} : \mathcal{X} \to \mathbb{R}_{+}$ that solve (7) and (8) in the main text.* The requirement takes the form of a system of bilinear inequalities that are linear in $\{\bar{u}(r)\}_{r \in \mathcal{X}}$ after conditioning on π_1^* and π_1^{**}.

The result below covers the general case. The test involves solving a system of bilinear inequalities in the variables $\bar{\pi}_s(\mathbf{x})$ (for all s and $\mathbf{x} \in \mathcal{L}$) and $\bar{u}(r)$ (for all $r \in \mathcal{X}$). Note that $\bar{\boldsymbol{\pi}}(\mathbf{x}) = (\bar{\pi}_1(\mathbf{x}), \bar{\pi}_2(\mathbf{x}), \ldots, \bar{\pi}_{\bar{s}}(\mathbf{x}))$ is used to construct the set of priors Π (in (A.2)) and that $\bar{\boldsymbol{\pi}}(\mathbf{x})$ is the distribution in Π that minimizes the expected utility of the bundle $\mathbf{x}$ (see (23)).

Proposition A.1. *A data set $\mathcal{O} = \{(\mathbf{x}^t, B^t)\}_{t=1}^{T}$ is MEU-rationalizable if and only if there is a function $\bar{\boldsymbol{\pi}} : \mathcal{L} \to \Delta_{++}$ and a strictly increasing function $\bar{u} : \mathcal{X} \to \mathbb{R}_{+}$ such that*

$$\bar{\boldsymbol{\pi}}(\mathbf{x}^t) \cdot \bar{\mathbf{u}}(\mathbf{x}^t) \geqslant \bar{\boldsymbol{\pi}}(\mathbf{x}) \cdot \bar{\mathbf{u}}(\mathbf{x}) \ \textit{ for all } \mathbf{x} \in \mathcal{L} \cap \underline{B}^t, \tag{A.3}$$

$$\bar{\boldsymbol{\pi}}(\mathbf{x}^t) \cdot \bar{\mathbf{u}}(\mathbf{x}^t) > \bar{\boldsymbol{\pi}}(\mathbf{x}) \cdot \bar{\mathbf{u}}(\mathbf{x}) \ \textit{ for all } \mathbf{x} \in \mathcal{L} \cap (\underline{B}^t \backslash \partial \underline{B}^t), \textit{ and} \tag{A.4}$$

$$\bar{\boldsymbol{\pi}}(\mathbf{x}) \cdot \bar{\mathbf{u}}(\mathbf{x}) \leqslant \bar{\boldsymbol{\pi}}(\mathbf{x}') \cdot \bar{\mathbf{u}}(\mathbf{x}) \ \textit{for all } (\mathbf{x}, \mathbf{x}') \in \mathcal{L} \times \mathcal{L}. \tag{A.5}$$

If these conditions hold, $\mathcal{O}$ admits an MEU-rationalization where Π (in (A.2)) is the convex hull of $\{\bar{\boldsymbol{\pi}}(\mathbf{x})\}_{\mathbf{x}\in\mathcal{L}}$, the Bernoulli function $u : \mathbb{R}_+ \to \mathbb{R}$ extends $\bar{u}$, and

$$V(\mathbf{x}) = \min_{\boldsymbol{\pi}\in\Pi}\{\boldsymbol{\pi} \cdot \bar{\mathbf{u}}(\mathbf{x})\} = \bar{\boldsymbol{\pi}}(\mathbf{x}) \cdot \bar{\mathbf{u}}(\mathbf{x}) \ \textit{for all } \mathbf{x} \in \mathcal{L}. \tag{23}$$

Proof: Suppose that $\mathcal{O}$ is rationalizable by ϕ as defined by (A.2). For any $\mathbf{x}$ in the finite lattice $\mathcal{L}$, let $\bar{\boldsymbol{\pi}}(\mathbf{x})$ be an element in $\arg\min_{\boldsymbol{\pi}\in\Pi} \boldsymbol{\pi} \cdot \mathbf{u}(\mathbf{x})$ and let $\bar{u}$ be the restriction of u to $\mathcal{X}$. Then it is clear that the conditions (A.3)–(A.5) hold.

Conversely, suppose that there is a function $\bar{\boldsymbol{\pi}}$ and a strictly increasing function $\bar{u}$ obeying the conditions (A.3)–(A.5). Define Π as the convex hull of $\{\bar{\boldsymbol{\pi}}(\mathbf{x}) : \mathbf{x} \in \mathcal{L}\}$; Π is a nonempty and convex subset of Δ_{++} and it is compact in $\mathbb{R}^{\bar{s}}$ since $\mathcal{L}$ is finite. Suppose that there exists $\mathbf{x} \in \mathcal{L}$ and $\boldsymbol{\pi} \in \Pi$ such that $\boldsymbol{\pi} \cdot \bar{\mathbf{u}}(\mathbf{x}) < \bar{\boldsymbol{\pi}}(\mathbf{x}) \cdot \bar{\mathbf{u}}(\mathbf{x})$. Since $\boldsymbol{\pi}$ is a convex combination of elements in $\{\bar{\boldsymbol{\pi}}(\mathbf{x}) : \mathbf{x} \in \mathcal{L}\}$, there must exist $\mathbf{x}' \in \mathcal{L}$ such that $\bar{\boldsymbol{\pi}}(\mathbf{x}') \cdot \bar{\mathbf{u}}(\mathbf{x}) < \bar{\boldsymbol{\pi}}(\mathbf{x}) \cdot \bar{\mathbf{u}}(\mathbf{x})$, which contradicts (A.5). We conclude that $\bar{\boldsymbol{\pi}}(\mathbf{x}) \cdot \bar{\mathbf{u}}(\mathbf{x}) = \min_{\boldsymbol{\pi}\in\Pi} \boldsymbol{\pi} \cdot \bar{\mathbf{u}}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{L}$. We define $\phi : \mathbb{R}^{\bar{s}}_+ \to \mathbb{R}$ by $\phi(\mathbf{u}) = \min_{\boldsymbol{\pi}\in\Pi} \boldsymbol{\pi} \cdot \mathbf{u}$. Then the conditions (A.3) and (A.4) are just versions of (7) and (8) in the main text, and so Theorem 1 in the main text guarantees that there is Bernoulli function $u : \mathbb{R}_+ \to \mathbb{R}_+$ extending $\bar{u}$ such that $\mathcal{O}$ is rationalizable by $V(\mathbf{x}) = \phi(\mathbf{u}(\mathbf{x}))$. **QED**

A2.3 Variational preferences

A popular model of decision making under uncertainty which generalizes maxmin expected utility is variational preferences (VP), introduced by Maccheroni, Marinacci, and Rustichini (2006). In this model, a bundle $\mathbf{x} \in \mathbb{R}^{\bar{s}}_+$ has utility $V(\mathbf{x}) = \phi(\mathbf{u}(\mathbf{x}))$, where

$$\phi(\mathbf{u}) = \min_{\boldsymbol{\pi}\in\Delta_{++}} \{\boldsymbol{\pi} \cdot \mathbf{u} + c(\boldsymbol{\pi})\} \tag{A.6}$$

and $c : \Delta_{++} \to \mathbb{R}_+$ is a continuous and convex function with the following boundary condition: for any sequence $\boldsymbol{\pi}^n \in \Delta_{++}$ tending to $\tilde{\boldsymbol{\pi}}$, with $\tilde{\pi}_s = 0$ for some s, we obtain $c(\boldsymbol{\pi}^n) \to \infty$. This boundary condition, together with the continuity of c, guarantee that there is $\boldsymbol{\pi}^* \in \Delta_{++}$

that solves the minimization problem in (A.6).[2] Therefore, ϕ is well-defined and strictly increasing.

We say that $\mathcal{O} = \{(\mathbf{x}^t, B^t)\}_{t=1}^T$ is VP-rationalizable if there is a function ϕ in the collection Φ_{VP} of functions of the form (A.6), and a Bernoulli function $u : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for each t, $\phi(\mathbf{u}(\mathbf{x}^t), \boldsymbol{\pi}^t) \geqslant \phi(\mathbf{u}(\mathbf{x}), \boldsymbol{\pi}^t)$ for all $\mathbf{x} \in B^t$. By Theorem 1 in the main text, this holds if and only if there exists a function $c : \Delta_{++} \to \mathbb{R}_+$ that is continuous, convex, and has the boundary property, and an increasing function $\bar{u} : \mathcal{X} \to \mathbb{R}_+$ that together solve (7) and (8) in the main text, with ϕ defined by (A.6). The following result is a reformulation of this characterization that has a similar flavor to Proposition A.1; crucially, the necessary and sufficient conditions on $\mathcal{O}$ are formulated as a finite set of bilinear inequalities.

Proposition A.2. *A data set $\mathcal{O} = \{(\mathbf{x}^t, B^t)\}_{t=1}^T$ is VP-rationalizable if and only if there is a function $\bar{\boldsymbol{\pi}} : \mathcal{L} \to \Delta_{++}$, a function $\bar{c} : \mathcal{L} \to \mathbb{R}_+$, and a strictly increasing function $\bar{u} : \mathcal{X} \to \mathbb{R}_+$ such that*

$$\bar{\boldsymbol{\pi}}(\mathbf{x}^t) \cdot \bar{\mathbf{u}}(\mathbf{x}^t) + \bar{c}(\mathbf{x}^t) \geqslant \bar{\boldsymbol{\pi}}(\mathbf{x}) \cdot \bar{\mathbf{u}}(\mathbf{x}) + \bar{c}(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathcal{L} \cap \underline{B}^t, \tag{A.7}$$

$$\bar{\boldsymbol{\pi}}(\mathbf{x}^t) \cdot \bar{\mathbf{u}}(\mathbf{x}^t) + \bar{c}(\mathbf{x}^t) > \bar{\boldsymbol{\pi}}(\mathbf{x}) \cdot \bar{\mathbf{u}}(\mathbf{x}) + \bar{c}(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathcal{L} \cap (\underline{B}^t \backslash \partial \underline{B}^t), \text{ and} \tag{A.8}$$

$$\bar{\boldsymbol{\pi}}(\mathbf{x}) \cdot \bar{\mathbf{u}}(\mathbf{x}) + \bar{c}(\mathbf{x}) \leqslant \bar{\boldsymbol{\pi}}(\mathbf{x}') \cdot \bar{\mathbf{u}}(\mathbf{x}) + \bar{c}(\mathbf{x}') \text{ for all } (\mathbf{x}, \mathbf{x}') \in \mathcal{L} \times \mathcal{L}. \tag{A.9}$$

If these conditions hold, then $\mathcal{O}$ can be rationalized by a variational preference V, with ϕ given by (A.6), such that the following holds:

(i) *$c : \Delta_{++} \to \mathbb{R}_+$ satisfies $c(\bar{\boldsymbol{\pi}}(\mathbf{x})) = \bar{c}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{L}$;*

(ii) *the Bernoulli function $u : \mathbb{R}_+ \to \mathbb{R}$ satisfies $\bar{u}(r) = u(r)$ for all $r \in \mathcal{X}$; and*

(iii) *$\bar{\boldsymbol{\pi}}(\mathbf{x}) \in \arg\min_{\boldsymbol{\pi} \in \Delta_{++}} \{\boldsymbol{\pi} \cdot \mathbf{u}(\mathbf{x}) + c(\boldsymbol{\pi})\}$, leading to $V(\mathbf{x}) = \bar{\boldsymbol{\pi}}(\mathbf{x}) \cdot \bar{\mathbf{u}}(\mathbf{x}) + \bar{c}(\mathbf{x})$, for all $\mathbf{x} \in \mathcal{L}$.*

Proof: Suppose $\mathcal{O}$ is rationalizable by ϕ as defined by (A.6). Let $\bar{u}$ be the restriction of u to $\mathcal{X}$. For any $\mathbf{x}$ in $\mathcal{L}$, let $\bar{\boldsymbol{\pi}}(\mathbf{x})$ be an element in $\arg\min_{\boldsymbol{\pi} \in \Delta_{++}} \{\boldsymbol{\pi} \cdot \mathbf{u}(\mathbf{x}) + c(\boldsymbol{\pi})\}$, and let $\bar{c}(\mathbf{x}) = c(\bar{\boldsymbol{\pi}}(\mathbf{x}))$. Then it is clear that the conditions (A.7)–(A.9) hold.

[2] Indeed, pick any $\tilde{\boldsymbol{\pi}} \in \Delta_{++}$ and define $S = \{\boldsymbol{\pi} \in \Delta_{++} : \boldsymbol{\pi} \cdot \mathbf{u} + c(\boldsymbol{\pi}) \leqslant \tilde{\boldsymbol{\pi}} \cdot \mathbf{u} + c(\tilde{\boldsymbol{\pi}})\}$. The boundary condition and continuity of c guarantee that S is compact in $\mathbb{R}^{\bar{s}}$ and hence $\arg\min_{\boldsymbol{\pi} \in S} \{\boldsymbol{\pi} \cdot \mathbf{u} + c(\boldsymbol{\pi})\} = \arg\min_{\boldsymbol{\pi} \in \Delta_{++}} \{\boldsymbol{\pi} \cdot \mathbf{u} + c(\boldsymbol{\pi})\}$ is nonempty.

Conversely, suppose that there is a strictly increasing function $\bar{u}$ and functions $\bar{\boldsymbol{\pi}}$ and $\bar{c}$ obeying conditions (A.7)–(A.9). For every $\boldsymbol{\pi} \in \Delta_{++}$, define $\tilde{c}(\boldsymbol{\pi}) = \max_{\mathbf{x}\in\mathcal{L}}\{\bar{c}(\mathbf{x}) - (\boldsymbol{\pi} - \bar{\boldsymbol{\pi}}(\mathbf{x})) \cdot \bar{\mathbf{u}}(\mathbf{x})\}$. It follows from (A.9) that $\bar{c}(\mathbf{x}') \geqslant \bar{c}(\mathbf{x}) - (\bar{\boldsymbol{\pi}}(\mathbf{x}') - \bar{\boldsymbol{\pi}}(\mathbf{x})) \cdot \bar{\mathbf{u}}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{L}$. Therefore, $\tilde{c}(\bar{\boldsymbol{\pi}}(\mathbf{x}')) = \bar{c}(\mathbf{x}')$ for any $\mathbf{x}' \in \mathcal{L}$. The function $\tilde{c}$ is convex and continuous but it need not obey the boundary condition. However, we know there is a function c defined on Δ_{++} that is convex, continuous, obeys the boundary condition, with $c(\boldsymbol{\pi}) \geqslant \tilde{c}(\boldsymbol{\pi})$ for all $\boldsymbol{\pi} \in \Delta_{++}$ and $c(\boldsymbol{\pi}) = \tilde{c}(\boldsymbol{\pi})$ for $\boldsymbol{\pi} \in \{\bar{\boldsymbol{\pi}}(\mathbf{x}) : \mathbf{x} \in \mathcal{L}\}$. We claim that, with c so defined, $\min_{\boldsymbol{\pi}\in\Delta_{++}}\{\boldsymbol{\pi} \cdot \bar{\mathbf{u}}(\mathbf{x}) + c(\boldsymbol{\pi})\} = \boldsymbol{\pi}(\mathbf{x}) \cdot \bar{\mathbf{u}}(\mathbf{x}) + \bar{c}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{L}$. Indeed, for any $\boldsymbol{\pi} \in \Delta_{++}$,

$$\boldsymbol{\pi} \cdot \bar{\mathbf{u}}(\mathbf{x}) + c(\boldsymbol{\pi}) \geqslant \boldsymbol{\pi} \cdot \bar{\mathbf{u}}(\mathbf{x}) + \tilde{c}(\boldsymbol{\pi}) \geqslant \boldsymbol{\pi} \cdot \bar{\mathbf{u}}(\mathbf{x}) + \bar{c}(\mathbf{x}) - (\boldsymbol{\pi} - \bar{\boldsymbol{\pi}}(\mathbf{x})) \cdot \bar{\mathbf{u}}(\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x}) + \bar{c}(\mathbf{x}).$$

On the other hand, $\bar{\boldsymbol{\pi}}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) + c(\bar{\boldsymbol{\pi}}(\mathbf{x})) = \bar{\boldsymbol{\pi}}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) + \bar{c}(\mathbf{x})$, which establishes the claim. We define $\phi : \mathbb{R}^{\bar{s}}_{+} \to \mathbb{R}$ by (A.6); then (A.7) and (A.8) are just versions of (7) and (8) in the main text, and so Theorem 1 in the main text guarantees that there is a Bernoulli function $u : \mathbb{R}_{+} \to \mathbb{R}_{+}$ extending $\bar{u}$ such that $\mathcal{O}$ is rationalizable by $V(\mathbf{x}) = \phi(\mathbf{u}(\mathbf{x}))$. **QED**

A2.4 Models with budget-dependent reference points

So far in our discussion we have assumed that the agent has a preference over different contingent outcomes, without being too specific as to what actually constitutes an outcome in the agent's mind. On the other hand, models such as prospect theory have often emphasized the impact of reference points, and *changing* reference points, on decision making. Some of these phenomena can be easily accommodated within our framework.

For example, imagine an experiment in which subjects are asked to choose from a constraint set of state contingent monetary prizes. Assuming that there are $\bar{s}$ states and that the subject never suffers a loss, we can represent each prize by a vector $\mathbf{x} \in \mathbb{R}^{\bar{s}}_{+}$. The subject is observed to choose $\mathbf{x}^t$ from $B^t \subset \mathbb{R}^{\bar{s}}_{+}$, so the data set is $\mathcal{O} = \{(\mathbf{x}^t, B^t)\}_{t=1}^{T}$. The standard way of thinking about the subject's behavior is to assume his choice from B^t is governed by a preference defined on the prizes, which implies that the situation where he never receives a prize (formally the vector 0) is the subject's constant reference point. But a researcher may well be interested in whether the subject has a different reference point or multiple reference points that vary with the budget (and perhaps manipulable by the researcher). Most obviously, suppose that the subject has an endowment point $\boldsymbol{\omega}^t \in \mathbb{R}^{\bar{s}}_{+}$ and a classical budget set

$B^t = \{\mathbf{x} \in \mathbb{R}^{\bar{s}}_+ : \mathbf{p}^t \cdot \mathbf{x} \leqslant \mathbf{p}^t \cdot \boldsymbol{\omega}^t\}$. In this case, a possible hypothesis is that the subject will evaluate different bundles in B^t based on a utility function defined on the deviation from the endowment; in other words, the endowment is the subject's reference point. Another possible reference point is that bundle in B^t which gives the same payoff in every state.

Whatever it may be, suppose the researcher has a hypothesis about the possible reference point at observation t, which we shall denote by $\mathbf{e}^t \in \mathbb{R}^{\bar{s}}_+$, and that the subject chooses according to some utility function $V : [-K, \infty)^{\bar{s}} \to \mathbb{R}_+$ where $K > 0$ is sufficiently large so that $[-K, \infty)^{\bar{s}} \subset \mathbb{R}^{\bar{s}}$ contains all the possible reference point-dependent outcomes in the data, i.e., the set $\bigcup_{t=1}^T \tilde{B}^t$, where

$$\tilde{B}^t = \{\mathbf{x}' \in \mathbb{R}^{\bar{s}} : \ \mathbf{x}' = \mathbf{x} - \mathbf{e}^t \text{ for some } \mathbf{x} \in B^t\}.$$

Let $\{\phi(\cdot, t)\}_{t=1}^T$ be a collection of functions, where $\phi(\cdot, t) : [-K, \infty)^{\bar{s}} \to \mathbb{R}$ is increasing in all of its arguments. We say that $\mathcal{O} = \{(\mathbf{x}^t, B^t)\}_{t=1}^T$ is *rationalizable by* $\{\phi(\cdot, t)\}_{t=1}^T$ *and the reference points* $\{\mathbf{e}^t\}_{t=1}^T$ if there exists a Bernoulli function $u : [-K, \infty) \to \mathbb{R}_+$ such that $\phi(\mathbf{u}(\mathbf{x}^t - \mathbf{e}^t), t) \geqslant \phi(\mathbf{u}(\mathbf{x} - \mathbf{e}^t), t)$ for all $\mathbf{x} \in B^t$. This is formally equivalent to saying that the modified data set $\mathcal{O}' = \{(\mathbf{x}^t - \mathbf{e}^t, \tilde{B}^t)\}_{t=1}^T$ is rationalizable by $\{\phi(\cdot, t)\}_{t=1}^T$. Applying Theorem 1 in the main text, rationalizability holds if and only if there is a strictly increasing function $\bar{u} : \mathcal{X} \to \mathbb{R}_+$ that obeys (7) and (8) in the main text, where

$$\mathcal{X} = \{r \in \mathbb{R} : r = x^t_s - e^t_s \text{ for some } t,\ s\} \cup \{-K\}.$$

Therefore, we may test whether $\mathcal{O}$ is rationalizable by expected utility, or by any of the models described so far, in conjunction with budget dependent reference points. Note that a test of rank dependent utility in this context is sufficiently flexible to accommodate phenomena emphasized by cumulative prospect theory (see Tversky and Kahneman (1992)), such as a Bernoulli function $u : [-K, \infty) \to \mathbb{R}$ that is S-shaped (and hence nonconcave) around 0 and probabilities distorted by a weighting function.

A3. Relationship between the SMU and MVU models

The following proposition re-states and proves the claims made in Section 4 of the main text regarding the relationship between the stochastically monotone utility (SMU) and mean-variance utility (MVU) models.

Proposition A.3. (i) *When there are two states, any data set that is MVU-rationalizable at some efficiency index e is SMU-rationalizable at the same efficency index.* (ii) *When there are two equiprobable states, any data set that is SMU-rationalizable at some efficiency index e and does not contain an observation where $p_1^t = p_2^t$ is also MVU-rationalizable at the same efficiency index.*

Proof: *(i)* It suffices to show that the presence of an SMU-revealed preference cycle containing a strict SMU-revealed preference relation would imply the presence of an MVU-revealed preference cycle with a strict MVU-revealed preference relation, so that a data set that is not SMU-rationalizable is also not MVU-rationalizable. Suppose that there is an SMU-revealed preference cycle of the form given by (16), with $\succcurlyeq_e^*$ and $\succ_e^*$ interpreted as the SMU-revealed preference and SMU-revealed strict preference relations respectively. We claim that we can always choose the cycle so that

$$\text{if } \mathbf{x}^{t_{i-1}} \succ_e^* \mathbf{x}^{t_i} \text{ then } \mathbf{x}^{t_{i-1}} \not> \mathbf{x}^{t_i}. \tag{A.9}$$

This is because, should $\mathbf{x}^{t_{i-1}} > \mathbf{x}^{t_i}$, then it simply follows from the definition of the SMU-revealed preference relations that $\mathbf{x}^{t_{i-2}} \succ_e^* \mathbf{x}^{t_i}$. So we have eliminated $\mathbf{x}^{t_{i-1}}$ from the cycle. Repeating this procedure if necessary, we eventually end up with a cycle containing at least one SMU-revealed strict preference and with every revealed strict preference obeying (A.9). We claim that in this cycle, $\mathbf{x}^{t_{i-1}}$ is MVU-revealed preferred (MVU-revealed strictly preferred) to $\mathbf{x}^{t_i}$ whenever $\mathbf{x}^{t_{i-1}}$ is MVU-revealed preferred (MVU-revealed strictly preferred) to $\mathbf{x}^{t_i}$. Consequently, we obtain an MVU-revealed preference cycle containing a strict MVU-revealed preference relation. We consider in turn the cases where $\pi_1 = \pi_2$ and $\pi_1 < \pi_2$.

Suppose that $\pi_1 = \pi_2$ and $\mathbf{x}^t$ is SMU-revealed preferred to $\mathbf{x}^{t'}$. If $\mathbf{x}^{t'} \in B^t(e)$, then obviously $\mathbf{x}^t$ is also MVU-revealed preferred to $\mathbf{x}^{t'}$. If $\mathbf{x}^{t'} \notin B^t(e)$, then $\tilde{\mathbf{x}}^{t'}$ (the reflection of $\mathbf{x}^{t'}$ with respect to the 45 degree line) is in $B^t(e)$; since $\tilde{\mathbf{x}}^{t'}$ has the same mean and variance as $\mathbf{x}^{t'}$, so $\mathbf{x}^t$ is also MVU-revealed preferred to $\mathbf{x}^{t'}$ at threshold e. Now suppose $\mathbf{x}^t$ is SMU-revealed strictly preferred to $\mathbf{x}^{t'}$ and $\mathbf{x}^t \not> \mathbf{x}^{t'}$. Then either $\mathbf{p}^t \cdot \mathbf{x}^{t'} < e\,\mathbf{p}^t \cdot \mathbf{x}^t$ or $\mathbf{p}^t \cdot \tilde{\mathbf{x}}^{t'} < e\,\mathbf{p}^t \cdot \mathbf{x}^t$;[3] in either case $\mathbf{x}^t$ is MVU-revealed strictly preferred to $\mathbf{x}^{t'}$.

Consider now the case where $\pi_1 < \pi_2$. Suppose $\mathbf{x}^t$ is SMU-revealed preferred to $\mathbf{x}^{t'}$. If

[3] Obviously, if $\mathbf{x}^t > \mathbf{x}^{t'}$ then neither condition is necessary.

$\mathbf{x}^{t'} \in B^t(e)$ then clearly $\mathbf{x}^t$ is MVU-revealed preferred to $\mathbf{x}^{t'}$. Consider now the case where $x_1^{t'} > x_2^{t'}$ and $\tilde{\mathbf{x}}^{t'} \in B^t(e)$. The locus of bundles with the same mean as $\mathbf{x}^{t'}$ (which obviously passes through $\mathbf{x}^{t'}$ itself) is flatter than the line orthogonal to the 45 degree line and passing through $\mathbf{x}^{t'}$; furthermore, one could check that the unique bundle $\mathbf{a} = (a_1, a_2)$ with $a_2 > a_1$ and the same mean and variance as $\mathbf{x}^{t'}$ has the property that $\mathbf{a} < \tilde{\mathbf{x}}^{t'}$. So $\mathbf{a} \in B^t(e)$ and in fact $\mathbf{p}^t \cdot \mathbf{a} < e\,\mathbf{p}^t \cdot \mathbf{x}^t$, from which we conclude that $\mathbf{x}^t$ is MVU-strictly revealed preferred to $\mathbf{x}^{t'}$. Now suppose $\mathbf{x}^t$ is SMU-revealed strictly preferred to $\mathbf{x}^{t'}$, with $\mathbf{x}^t \not\geq \mathbf{x}^{t'}$. Then either $\mathbf{x}^t$ satisfies $\mathbf{p}^t \cdot \mathbf{x}^{t'} < e\,\mathbf{p}^t \cdot \mathbf{x}^t$ or $x_1^{t'} > x_2^{t'}$ and $\mathbf{p}^t \cdot \tilde{\mathbf{x}}^{t'} \leqslant e\,\mathbf{p}^t \cdot \mathbf{x}^t$. In either case we conclude that $\mathbf{x}^t$ is MVU-strictly revealed preferred to $\mathbf{x}^{t'}$.

(ii) It suffices to show that the presence of an MVU-revealed preference cycle containing a strict MVU-revealed preference relation would imply the presence of an SMU-revealed preference cycle with a strict SMU-revealed preference relation, so that a data set is not SMU-rationalizable when it is not MVU-rationalizable. Suppose that there is an MVU-revealed preference cycle of the form given by (16), with $\succcurlyeq_e^*$ and $\succ_e^*$ interpreted as the MVU-revealed preference and MVU-revealed strict preference respectively. We claim that we can always choose the cycle so that we do not have $(\mu(\mathbf{x}^{t_{i-1}}), -\sigma(\mathbf{x}^{t_{i-1}})) \geqslant (\mu(\mathbf{x}^{t_i}), -\sigma(\mathbf{x}^{t_i}))$. Indeed, if this occurs, we can always remove $\mathbf{x}^{t_{i-1}}$ from the cycle, since it follows from the definition of the MVU-revealed preference relations that $\mathbf{x}^{t_{i-2}} \succcurlyeq_e^* \mathbf{x}^{t_i}$ and $\mathbf{x}^{t_{i-2}} \succ_e^* \mathbf{x}^{t_i}$ if either $\mathbf{x}^{t_{i-2}} \succ_e^* \mathbf{x}^{t_{i-1}}$ or $(\mu(\mathbf{x}^{t_{i-1}}), -\sigma(\mathbf{x}^{t_{i-1}})) > (\mu(\mathbf{x}^{t_i}), -\sigma(\mathbf{x}^{t_i}))$. Repeating this removal procedure if necessary we will have eventually have an MVU-revealed preference cycle containing a strict MVU-revealed preference relation such that $(\mu(\mathbf{x}^{t_{i-1}}), -\sigma(\mathbf{x}^{t_{i-1}})) \not\geqslant (\mu(\mathbf{x}^{t_i}), -\sigma(\mathbf{x}^{t_i}))$ at each link in the cycle. We claim that, on such a cycle, the MVU-revealed preference relations imply the SMU-revealed preference relations at the same threshold.

We maintain the assumption that $(\mu(\mathbf{x}^t), -\sigma(\mathbf{x}^t)) \not\geqslant (\mu(\mathbf{x}^{t'}), -\sigma(\mathbf{x}^{t'}))$. If $\mathbf{x}^t$ is MVU-revealed preferred to $\mathbf{x}^{t'}$ at threshold e, then $\mathbf{x}^t$ is SMU-revealed preferred to $\mathbf{x}^{t'}$ at the same threshold if $\mathbf{p}^t \cdot \mathbf{x}^{t'} \leqslant e\,\mathbf{p}^t \cdot \mathbf{x}^t$. If not, there is a bundle $\mathbf{y}$ satisfying $\mathbf{p}^t \cdot \mathbf{y} \leqslant e\,\mathbf{p}^t \cdot \mathbf{x}^t$ such that $(\mu(\mathbf{y}), -\sigma(\mathbf{y})) \geqslant ((\mu(\mathbf{x}^{t'}), -\sigma(\mathbf{x}^{t'}))$. It is straightforward to check that when this occurs, the bundle $\tilde{\mathbf{x}}^{t'}$ (which has the same mean and variance as $\mathbf{x}^{t'}$) must also satisfy $\mathbf{p}^t \cdot \tilde{\mathbf{x}}^{t'} \leqslant e\,\mathbf{p}^t \cdot \mathbf{x}^t$ and hence $\mathbf{x}^t$ is SMU-revealed preferred to $\mathbf{x}^{t'}$ at threshold e.[4] Now suppose $\mathbf{x}^t$ is MVU-

[4] For any bundle $\mathbf{z}$, the set of bundles with the same mean lie on a straight line passing through $\mathbf{z}$ and orthogonal to the 45 degree line, with bundles further away from the 45 degree line having a higher variance.

strictly revealed preferred to $\mathbf{x}^{t'}$. If $\mathbf{p}^t \cdot \mathbf{x}^t < e\,\mathbf{p}^t \cdot \mathbf{x}^t$, then clearly $\mathbf{x}^t$ is SMU-strictly revealed preferred to $\mathbf{x}^t$. If $\mathbf{p}^t \cdot \mathbf{x}^t \geqslant e\,\mathbf{p}^t \cdot \mathbf{x}^t$, then one could check that $\mathbf{p}^t \cdot \tilde{\mathbf{x}}^t < e\,\mathbf{p}^t \cdot \mathbf{x}^t$ so long as $p_1^t \neq p_2^t$, which again implies that $\mathbf{x}^t$ is SMU-strictly revealed preferred to $\mathbf{x}^t$.[5] **QED**

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Note also that two bundles equidistant from the 45 degree line have the same variance, even if they have different means.

[5] If $p_1^t = p_2^t$, it is possible for $\mathbf{x}^t$ to be MVU-strictly revealed preferred to $\mathbf{x}^{t'}$ such that $(\mu(\mathbf{x}^t, -\sigma(\mathbf{x}^t) \not\geqslant (\mu(\mathbf{x}^{t'}, -\sigma(\mathbf{x}^{t'})$, $\mathbf{p}^t \cdot \mathbf{x}^{t'} = e\,\mathbf{p}^t \cdot \mathbf{x}^t$ and $\mathbf{p}^t \cdot \tilde{\mathbf{x}}^{t'} = e\,\mathbf{p}^t \cdot \mathbf{x}^t$.

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